Lattice for Supersymmetric Physics

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Lattice for Beyond the Standard Model Physics
Lawrence Livermore National Laboratory, 25 April 2015

arXiv:1405.0644, arXiv:1410.6971, arXiv:1411.0166 & more to come
with Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Context: Why lattice supersymmetry

Lattice discretization provides non-perturbative,
gauge-invariant regularization of vectorlike gauge theories

We've discussed many ways lattice studies
can improve our knowledge of strongly coupled field theories

We can imagine many potential susy applications, including

- Compute Wilson loops, spectrum, scaling dimensions, etc.,
  complementing perturbation theory, holography, bootstrap, ...
- Further direct checks of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine AdS/CFT-based modelling
  (e.g., QCD phase diagram, condensed matter systems)

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(e.g., QCD phase diagram, condensed matter systems)

Many ideas probably infeasible; relatively few have been explored

Context: Why not lattice supersymmetry

There is a problem with supersymmetry in discrete space-time

Recall supersymmetry extends Poincaré symmetry

by spinorial generators Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ with $\mathrm{I} = 1, \cdots, \mathcal{N}$

The resulting algebra includes $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$

P_μ generates infinitesimal translations, which don't exist on the lattice

$\Longrightarrow$ supersymmetry explicitly broken at classical level

Explicitly broken supersymmetry $\Longrightarrow$ relevant susy-violating operators (typically many)

Fine-tuning their couplings to restore supersymmetry is generally not practical in numerical lattice calculations

Special cases in four dimensions

Minimal ($\mathcal{N} = 1$) supersymmetric Yang–Mills

SU(N) gauge theory with massless gaugino in adjoint rep.

No scalar fields

$\Longrightarrow$ gaugino mass is only relevant susy-violating operator
$\Longrightarrow$ chiral lattice fermions (overlap / domain wall) protect susy

Scalar fields (from matter multiplets or non-minimal susy)
introduce many more relevant susy-violating operators

In this case (some subset of) the susy algebra must be preserved
to permit practical lattice calculations

Maximal ($\mathcal{N} = 4$) supersymmetric Yang–Mills (SYM)

The only known 4d system with a supersymmetric lattice formulation

Remainder of talk will focus on recent progress with lattice $\mathcal{N} = 4$ SYM

Exact susy on the lattice: $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is a particularly interesting theory

—Context for development of AdS/CFT correspondence

—Testing ground for reformulations of scattering amplitudes

—Arguably simplest non-trivial field theory in four dimensions

Basic features:

- SU(*N*) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ}, all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms with coefficients related by symmetries
- Supersymmetric: 16 supercharges Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ with $\mathrm{I} = 1, \cdots, 4$
  Fields and *Q*'s transform under global SU(4) $\simeq$ SO(6) R symmetry
- Conformal: β function is zero for any 't Hooft coupling λ

Exact susy on the lattice: topological twisting

What is special about $\mathcal{N} = 4$ SYM

The 16 fermionic supercharges $Q^{\rm I}_\alpha$ and $\overline{Q}^{\rm I}_{\dot\alpha}$ fill a Kähler–Dirac multiplet:

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \gamma_\mu \mathcal{Q}_\mu + \gamma_\mu\gamma_\nu \mathcal{Q}_{\mu\nu} + \gamma_\mu\gamma_5 \mathcal{Q}_{\mu\nu\rho} + \gamma_5 \mathcal{Q}_{\mu\nu\rho\sigma}$$

$$\longrightarrow \mathcal{Q} + \gamma_a \mathcal{Q}_a + \gamma_a\gamma_b \mathcal{Q}_{ab}$$

with $a, b = 1, \cdots, 5$

This is a decomposition in representations of a "twisted rotation group"

$$\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\rm euc} \otimes \text{SO}(4)_R\right] \qquad\qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

This change of variables gives a susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

This subalgebra can be exactly preserved on the lattice

Twisted $\mathcal{N} = 4$ SYM

Everything transforms with **integer spin** under $\text{SO}(4)_{tw}$ — **no spinors**

$$Q^{\text{I}}_{\alpha} \text{ and } \overline{Q}^{\text{I}}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$
$$\Psi^{\text{I}} \text{ and } \overline{\Psi}^{\text{I}} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$
$$A_\mu \text{ and } \Phi^{\text{IJ}} \longrightarrow \mathcal{A}_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi}) \text{ and } \overline{\mathcal{A}}_a$$

The twisted-scalar supersymmetry $\mathcal{Q}$ acts as

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$
$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$
$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

↖ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

1. $\mathcal{Q}$ directly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f.
2. The susy subalgebra $\mathcal{Q}^2 \cdot = 0$ is manifest

Lattice $\mathcal{N} = 4$ SYM

The lattice theory is very nearly a direct transcription

- Covariant derivatives $\longrightarrow$ finite difference operators
- Gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

- Naive lattice action retains same form as continuum action and remains supersymmetric, $\mathcal{Q}S = 0$

Geometrical formulation facilitates discretization

η live on lattice sites $\qquad$ ψ_a live on links

χ_{ab} connect opposite corners of oriented plaquettes

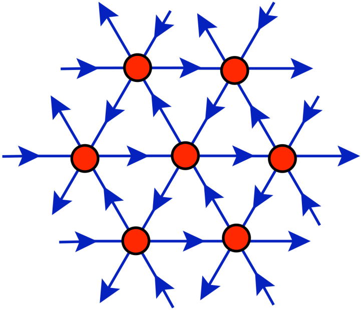
Orbifolding / dimensional deconstruction produces same lattice system

Five links in four dimensions $\longrightarrow A_4^*$ lattice

—Can picture A_4^* lattice
as 4d analog of 2d triangular lattice

—Preserves S_5 point group symmetry

—Basis vectors are non-orthogonal
and linearly dependent



S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1}: \quad \mathcal{U}_a \longrightarrow A_\mu + iB_\mu, \quad \phi + i\overline{\phi}$$

$$\psi_a \longrightarrow \psi_\mu, \quad \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4}: \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \overline{\psi}_\mu$$

Twisted $\mathcal{N} = 4$ SYM on the A_4^* lattice

—We have exact gauge invariance

—We exactly preserve $\mathcal{Q}$, one of 16 supersymmetries

—The S_5 point group symmetry
provides twisted R & Lorentz symmetry in the continuum limit

The high degree of symmetry has important consequences

- Moduli space preserved to all orders of lattice perturbation theory
  $\longrightarrow$ no scalar potential induced by radiative corrections
- β function vanishes at one loop in lattice perturbation theory
- Real-space RG blocking transformations preserve $\mathcal{Q}$ and S_5
- Only one marginal tuning to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

The theory is **almost** suitable for practical numerical calculations...

Numerical complications

1. Complex gauge field $\Longrightarrow$ U(N) = SU(N) $\otimes$ U(1) gauge invariance
   U(1) sector decouples only in continuum limit

2. $\mathcal{Q}\,\mathcal{U}_a = \psi_a \Longrightarrow$ gauge links must be elements of algebra
   Resulting flat directions required by supersymmetric construction
   but must be lifted to ensure $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

We need to add two deformations to regulate flat directions

SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$

U(1) plaquette determinant $\sim G \sum_{a \neq b} \left(\det \mathcal{P}_{ab} - 1\right)$

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

↖ susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant (new development)

New development

(arXiv:1504.0____)

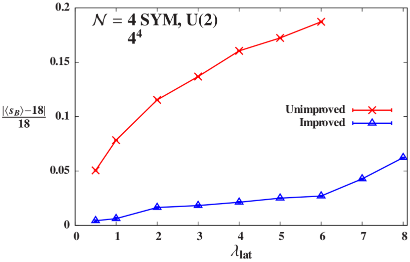
Scalar potential softly breaks $\mathcal{Q}$ supersymmetry

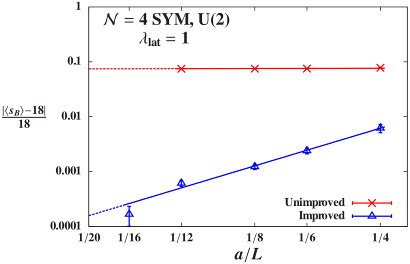
Plaquette determinant can be made $\mathcal{Q}$-invariant

Basic idea: Modify the equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \longrightarrow \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n) - 1\right]$$

Produces much smaller violations of $\mathcal{Q}$ Ward identity $\langle s_B\rangle = 9N^2/2$





Aside: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$
\begin{aligned}
S_{imp} &= S'_{exact} + S_{closed} + S'_{soft} \qquad (3.10)\\
S'_{exact} &= \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) \\
&\qquad + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det\mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{det}\\
S_{det} &= \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{b})\psi_a(n+\widehat{b})\right]\\
S_{closed} &= -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],\\
S'_{soft} &= \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}
$$

The lattice action is obviously very complicated

(For experts: $\gtrsim$100 inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

Physics result: Static potential is Coulombic at all λ

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)\,T}$

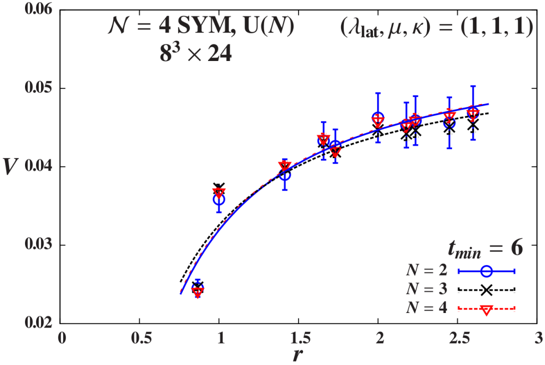
Fit $V(r)$ to Coulombic or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

C is Coulomb coefficient

σ is string tension



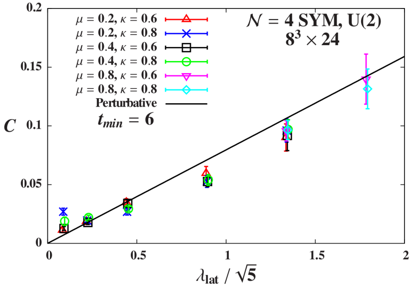
Fits to confining form always produce vanishing string tension $\sigma = 0$

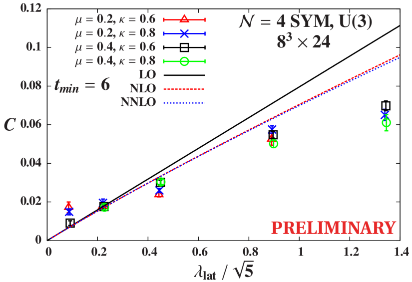
To be revisited with the improved action

Coupling dependence of Coulomb coefficient

Perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

AdS/CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$, $\lambda \to \infty$, $\lambda \ll N$





Left: Agreement with perturbation theory for $N = 2$, $\lambda \lesssim 2$

Right: Tantalizing $\sqrt{\lambda}$-like discrepancy for $N = 3$, $\lambda \gtrsim 1$

No visible dependence on (unimproved) soft $\mathcal{Q}$ breaking

Recapitulation

- Lattice supersymmetry is both enticing and challenging
- $\mathcal{N} = 4$ SYM is practical to study on the lattice
  thanks to exact preservation of susy subalgebra $\mathcal{Q}^2 = 0$
- The theory is simple; the lattice action is complicated
  $\longrightarrow$ Public code to reduce barriers to entry
- The static potential is always Coulombic
  For $N = 2$ $C(\lambda)$ is consistent with perturbation theory
  For $N = 3$ we may be seeing behavior predicted by AdS/CFT
- Many more directions are being — or can be — pursued
  - $\mathcal{N} = 4$ anomalous dimensions, e.g. for Konishi operator
  - Understanding the (absence of a) sign problem
  - Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, . . .

Thank you!

Thank you!

Collaborators

Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Funding and computing resources

Supplement: Konishi operator scaling dimension

$\mathcal{N} = 4$ SYM is conformal at any λ

All correlation functions decay algebraically $\propto r^{-\Delta}$

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) = Ar^{-2\Delta_K}$$

There are many predictions for the scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory, related to strong coupling by $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$ S duality
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Upper bounds from the conformal bootstrap program

We will add lattice gauge theory to this list

Konishi operator on the lattice

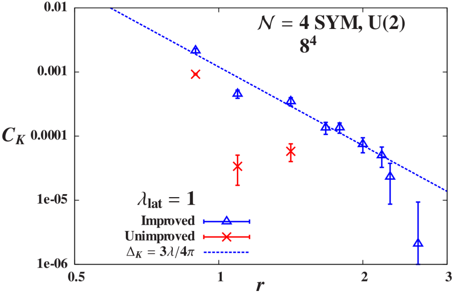
$$\mathcal{O}_K = \sum_{\rm I} \text{Tr}\left[\Phi^{\rm I}\Phi^{\rm I}\right] \longrightarrow \widehat{\mathcal{O}}_K = \sum_{a,\ b} \text{Tr}\left[\widehat{\varphi}^a\widehat{\varphi}^b\right]$$

$$\text{with } \widehat{\varphi}^a = \mathcal{U}_a\overline{\mathcal{U}}_a - \frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right]\mathbb{I}_N$$

$$\widehat{C}_K(r) = \widehat{\mathcal{O}}_K(x+r)\widehat{\mathcal{O}}_K(x) \propto r^{-2\Delta_K}$$

Need improved action
for reasonable $\widehat{C}_K(r)$ on 8^4 lattice

Improved results consistent with
power law using perturbative Δ_K



Fitting $\widehat{C}_K(r)$ is not a stable way to find Δ_K — we have better tools

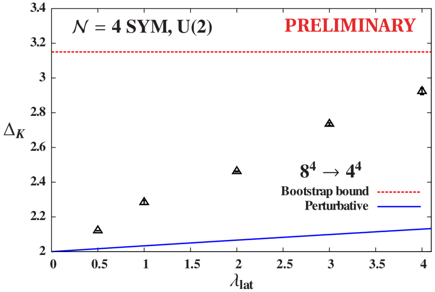
Preliminary Konishi Δ_K from Monte Carlo RG

Eigenvalues of MCRG stability matrix $\longrightarrow$ scaling dimensions

Simple trial (1×1 stability "matrix") correctly finds $\Delta_K \to 2$ as $\lambda \to 0$

Only statistical errors so far

Will check with independent finite-size scaling analysis



Many systematics to investigate

- ⋆ Larger volumes
- ⋆ More operators in stability matrix
- ⋆ (μ, G) dependence
- ⋆ More RG blocking steps
- ⋆ RG optimization
- ⋆ λ_{lat} renormalization

Supplement: The (absence of a) sign problem

In lattice gauge theory we compute operator expectation values

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int[d\mathcal{U}][d\overline{\mathcal{U}}]\,\mathcal{O}\,e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\,\text{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

$\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}|e^{i\alpha}$ can be complex for lattice $\mathcal{N}=4$ SYM
$\longrightarrow$ Complicates interpretation of $\left[e^{-S_B}\,\text{pf}\,\mathcal{D}\right]$ as Boltzmann weight

Have to **reweight** "phase-quenched" (pq) calculations

$$\langle\mathcal{O}\rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}}\int[d\mathcal{U}][d\overline{\mathcal{U}}]\,\mathcal{O}\,e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\,|\text{pf}\,\mathcal{D}| \qquad \langle\mathcal{O}\rangle = \frac{\left\langle\mathcal{O}e^{i\alpha}\right\rangle_{pq}}{\left\langle e^{i\alpha}\right\rangle_{pq}}$$

Sign problem: This breaks down if $\left\langle e^{i\alpha}\right\rangle_{pq}$ is consistent with zero

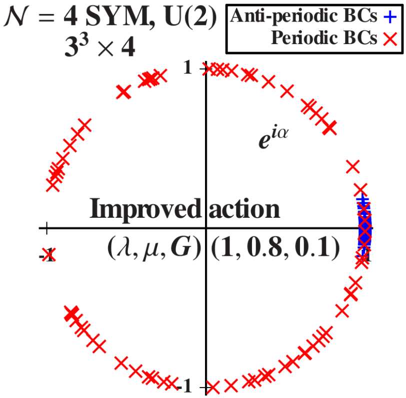
Illustration of sign problem and its absence

- With periodic temporal fermion boundary conditions
  we have an obvious sign problem, $\left\langle e^{i\alpha} \right\rangle_{pq}$ consistent with zero
- With anti-periodic BCs and all else the same $\left\langle e^{i\alpha} \right\rangle_{pq} \approx 1$
  $\longrightarrow$ phase reweighting not even necessary

Even stranger

Other $\langle \mathcal{O} \rangle_{pq}$ nearly identical
despite sign problem...

Can this be understood?



Pfaffian phase dependence on volume and N

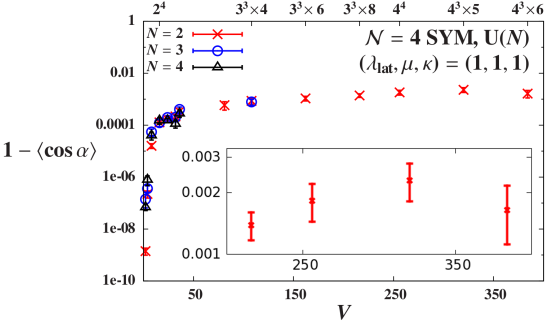
No indication of a sign problem with anti-periodic BCs

- $1 - \langle\cos(\alpha)\rangle \ll 1$ means $\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}|e^{i\alpha}$ nearly real and positive
- Fluctuations in pfaffian phase don't grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$
- To be revisited with the improved action

Hard calculations

Each $4^3 \times 6$ measurement required $\sim$8 days, $\sim$10GB memory

Parallel $\mathcal{O}(n^3)$ algorithm



Backup: Failure of Leibnitz rule in discrete space-time

Given that $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic,

why not try $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu$ for a discrete translation?

Here $\nabla_\mu\phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu\phi(x) + \frac{a}{2}\partial^2_\mu\phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned}\nabla_\mu\left[\phi(x)\chi(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\chi(x + a\widehat{\mu}) - \phi(x)\chi(x)\right] \\ &= \left[\nabla_\mu\phi(x)\right]\chi(x) + \phi(x)\nabla_\mu\chi(x) + a\left[\nabla_\mu\phi(x)\right]\nabla_\mu\chi(x)\end{aligned}$$

We only recover the Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$
$\Longrightarrow$ "Discrete supersymmetry" breaks down on the lattice

(Dondi & Nicolai, "Lattice Supersymmetry", 1977)

Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

The Kähler–Dirac representation is related to the usual $Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{I}}_{\dot\alpha}$ by

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \gamma_\mu \mathcal{Q}_\mu + \gamma_\mu \gamma_\nu \mathcal{Q}_{\mu\nu} + \gamma_\mu \gamma_5 \mathcal{Q}_{\mu\nu\rho} + \gamma_5 \mathcal{Q}_{\mu\nu\rho\sigma}$$
$$\longrightarrow \mathcal{Q} + \gamma_a \mathcal{Q}_a + \gamma_a \gamma_b \mathcal{Q}_{ab}$$
$$\text{with } a, b = 1, \cdots, 5$$

The 4×4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\right]$$

$\uparrow$ only $\text{SO}(4)_R \subset \text{SO}(6)_R$

Backup: A_4^* lattice with five links in four dimensions

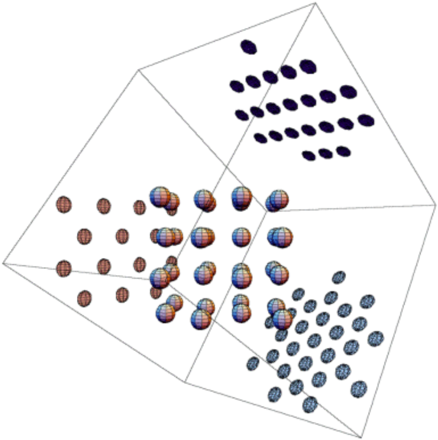
$A_a = (A_\mu, \phi)$ may remind you of dimensional reduction

On the lattice we want to treat all five $\mathcal{U}_a$ symmetrically
to obtain $S_5 \longrightarrow \text{SO}(4)_{tw}$ symmetry

—Start with hypercubic lattice
in 5d momentum space

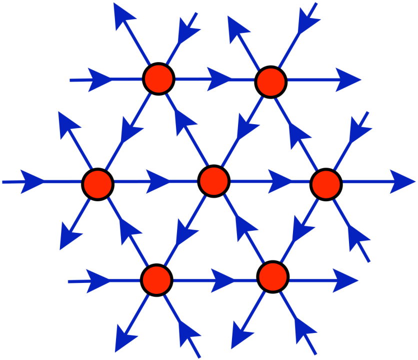
—**Symmetric** constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

—Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space



Backup: Hypercubic representation of A_4^* lattice

It is very convenient to represent the A_4^* lattice
as a hypercube with a backwards diagonal



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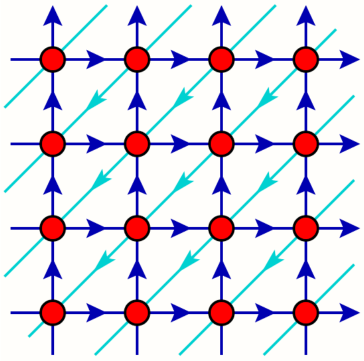


Backup: Twisted $\mathcal{N} = 4$ SYM action

Twisting gives manifestly supersymmetric action for $\mathcal{N} = 4$ SYM

$$S = \frac{N}{2\lambda_{\rm lat}} \mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{A}_a - \frac{1}{2}\eta d \right) - \frac{N}{8\lambda_{\rm lat}} \epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c \chi_{de}$$

$\mathcal{Q}S = 0$ follows from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

The lattice theory is very nearly a direct transcription

- Covariant derivatives $\longrightarrow$ finite difference operators
- Gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a$
- Lattice action retains same $\mathcal{Q}S = 0$ form as above

The unimproved action directly adds

$$S_{soft} = \frac{N}{2\lambda_{\rm lat}} \mu^2 \left(\frac{1}{N} \text{Tr} \left[\mathcal{U}_a \overline{\mathcal{U}}_a \right] - 1 \right)^2 + \kappa \left| \det \mathcal{P}_{ab} - 1 \right|^2$$

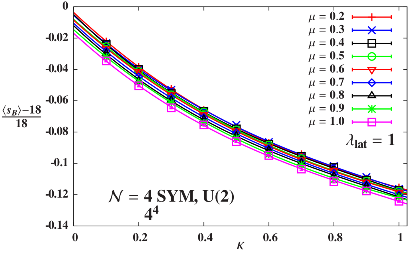
Both terms in S_{soft} softly break the $\mathcal{Q}$ supersymmetry

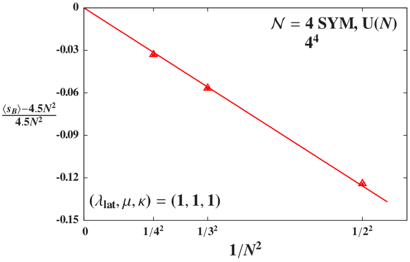
Backup: Unimproved soft susy breaking

Directly adding scalar potential and plaquette determinant to action explicitly breaks supersymmetry — det $\mathcal{P}_{ab}$ causes dominant effect

Left: The breaking is **soft** — guaranteed to vanish as $\mu, \kappa \longrightarrow 0$

Right: Soft $\mathcal{Q}$ breaking also suppressed $\propto 1/N^2$



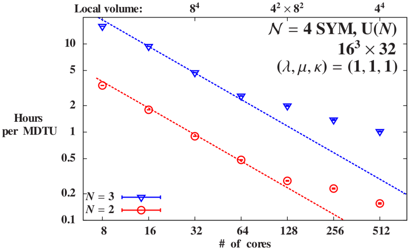


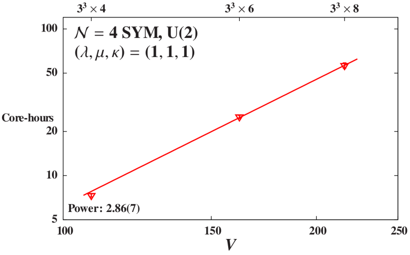
Backup: Code performance—weak and strong scaling

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(N_\Psi^3)$ pfaffian calculation (fixed local volume)
$N_\Psi \equiv 16N^2L^3N_T$ is number of fermion degrees of freedom

To be revisited with the improved action



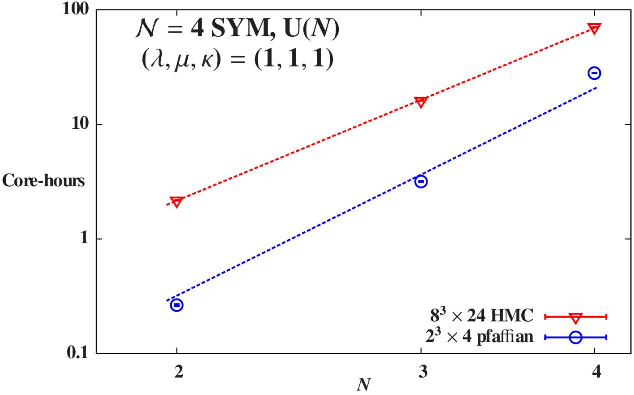


Both plots on log–log axes with power-law fits

Backup: Numerical costs for 2, 3 and 4 colors

Red: Find RHMC cost scaling $\sim N^5$ (recall adjoint fermion d.o.f. $\propto N^2$)

Blue: Pfaffian cost scaling consistent with expected N^6



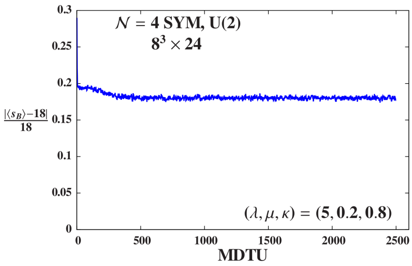
Backup: One problem with flat directions

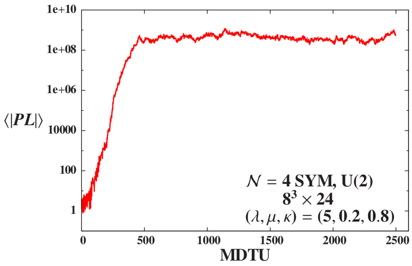
Gauge fields $\mathcal{U}_a$ can move far away from continuum form $\mathbb{I}_N + \mathcal{A}_a$ if $N\mu^2/(2\lambda_{\text{lat}})$ becomes too small

Example for two-color $(\lambda_{\text{lat}}, \mu, \kappa) = (5, 0.2, 0.8)$ on $8^3 \times 24$ volume

Left: Bosonic action is stable $\sim$18% off its supersymmetric value

Right: Polyakov loop wanders off to $\sim 10^9$

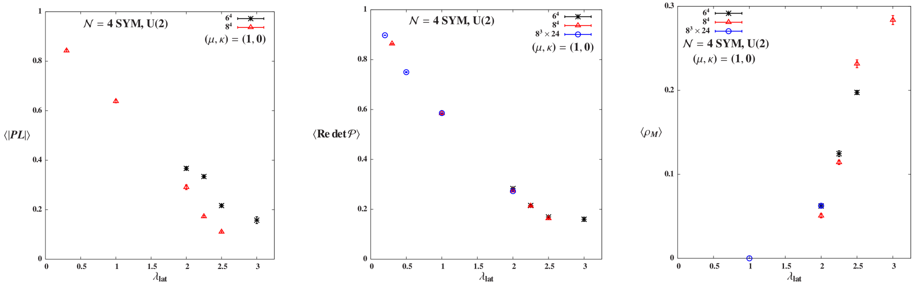




Backup: Another problem with U(1) flat directions

Flat directions in U(1) sector can induce transition to confined phase

This lattice artifact is not present in continuum $\mathcal{N} = 4$ SYM



Around the same $\lambda_{\text{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

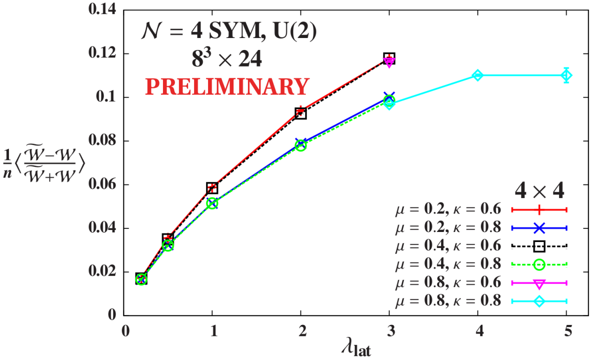
Right: Density of U(1) monopole world lines becomes non-zero

Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

Restoration of the other 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum limit follows from restoration of R symmetry (motivation for A_4^* lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

To be revisited with the improved action

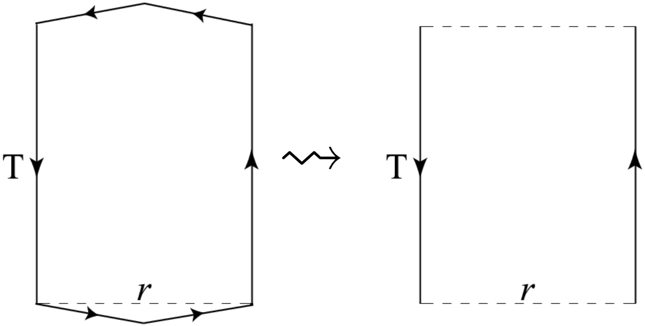


Backup: $\mathcal{N} = 4$ static potential from Wilson loops

Extract static potential $V(r)$ from $r \times T$ Wilson loops

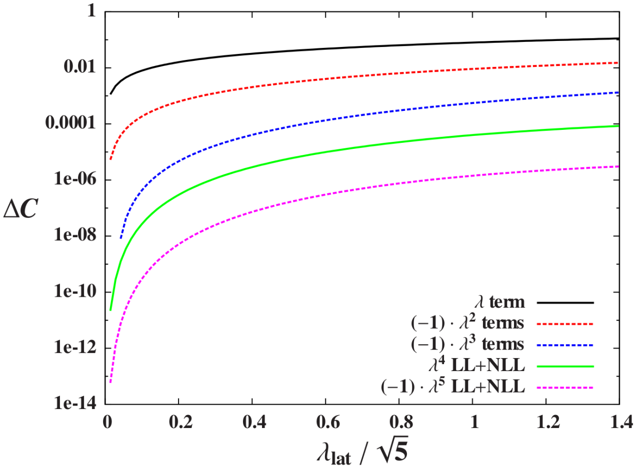
$$W(r, T) \propto e^{-V(r)\ T}$$

Coulomb gauge trick from lattice QCD reduces A_4^* lattice complications



Backup: Perturbation theory for Coulomb coefficient

For range of λ_{lat} currently being studied perturbation theory for the U(3) Coulomb coefficient appears well behaved

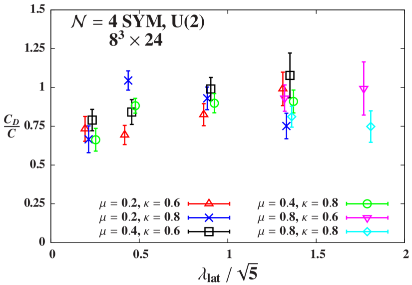


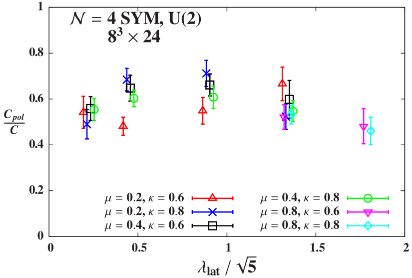
Backup: More tests of the U(2) static potential

Left: Projecting Wilson loops from U(2) $\longrightarrow$ SU(2)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 3/4$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





Both expected factors present, although (again) noisily

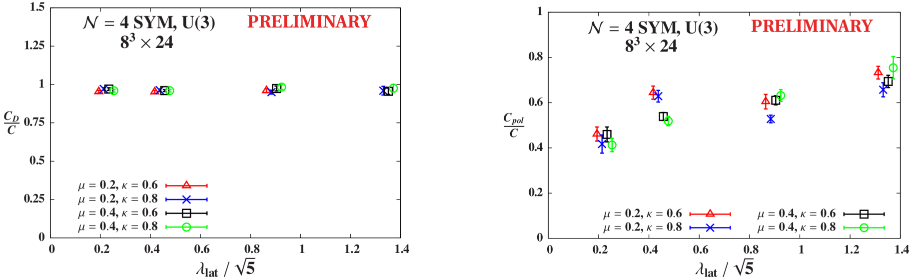
To be revisited with the improved action

Backup: More tests of the U(3) static potential

Left: Projecting Wilson loops from U(3) $\longrightarrow$ SU(3)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 8/9$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$

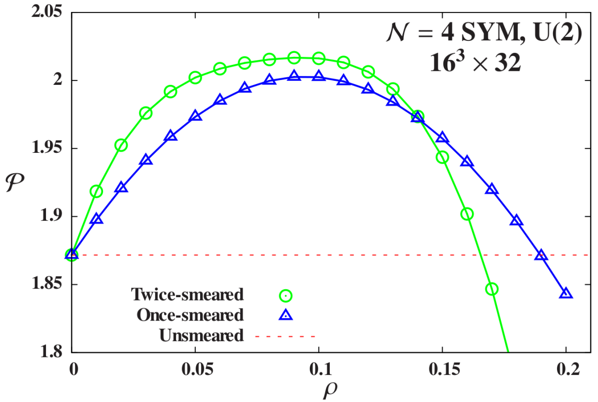


Ratios look slightly higher than expected, not as noisy as $N = 2$

To be revisited with the improved action

Backup: Smearing for noise reduction

Smearing may reduce noise in static potential (etc.) measurements
—Stout smearing implemented and tested
—APE or HYP (without unitary projection) may work better for Konishi



May be less important with the improved action

Backup: Konishi operator on the lattice

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \qquad\qquad \mathrm{I} = 1, \cdots, 6$$

On the lattice the scalars Φ^{I} are twisted
and wrapped up in the complexified gauge field $\mathcal{U}_a$

Given $\mathcal{U}_a \approx \mathbb{I}_N + \mathcal{A}_a$ the most obvious way to extract the scalars is

$$\widehat{\varphi}^a = \mathcal{U}_a\overline{\mathcal{U}}_a - \frac{1}{N}\mathrm{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right]\mathbb{I}$$

This is still twisted, so all $\{a, b\}$ contribute to R-singlet Konishi

$$\widehat{\mathcal{O}}_K = \sum_{a,\ b} \mathrm{Tr}\left[\widehat{\varphi}^a\widehat{\varphi}^b\right] \qquad\qquad a,\ b = 1, \cdots, 5$$

Backup: Scaling dimensions from Monte Carlo RG

Write system as (infinite) sum of operators $\mathcal{O}_i$ with couplings c_i

Couplings c_i flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Consider linear expansion around fixed point $H^\star$ with couplings $c_i^\star$

$$c_i^{(n)} - c_i^\star = \sum_j \left.\frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}}\right|_{H^\star} \left(c_j^{(n-1)} - c_j^\star\right) \equiv \sum_j T_{ij}^\star \left(c_j^{(n-1)} - c_j^\star\right)$$

$T_{ij}^\star$ is the stability matrix

Eigenvalues of $T_{ij}^\star \longrightarrow$ scaling dimensions of corresponding operators

Backup: Pfaffian phase dependence on λ_{lat}, μ, κ

Fluctuations in phase grow as λ_{lat} increases

but we observe little dependence on κ

To be revisited with the improved action