Chaotic Behavior in Matrix Gauge Theories

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In progress, with Masanori Hanada and Steve Shenker

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Overview

- Chaotic behavior of gauge theories using **real-time** dynamics
- Motivations:
  - Thermalization in quark-gluon plasma
  - Black hole dynamics
- Classical dynamics of 0+1D gauge theory with matrix degrees of freedom

The Gauge Theory

Matrix gauge theory

$$L = \frac{N}{2\lambda}\mathrm{Tr}\left(\sum_{i=1}^{9}(\dot{X}_i)^2 + \frac{1}{2}\sum_{i,j=1}^{9}[X_i, X_j]^2\right) + \cdots$$

gauge field,
fermions

- Quantum mechanics of N x N Hermitian matrices:

$$X_1(t), \ldots, X_9(t)$$

- $\mathrm{U}(N)$ gauge symmetry, $\mathrm{SO}(9)$ global symmetry

- Large N limit: keep $\lambda = g_{\mathrm{YM}}^2 N$ fixed

Holographic dualities

Supersymmetric Yang-Mills in 3+1D ⟷ 9+1D Gravity (string theory)

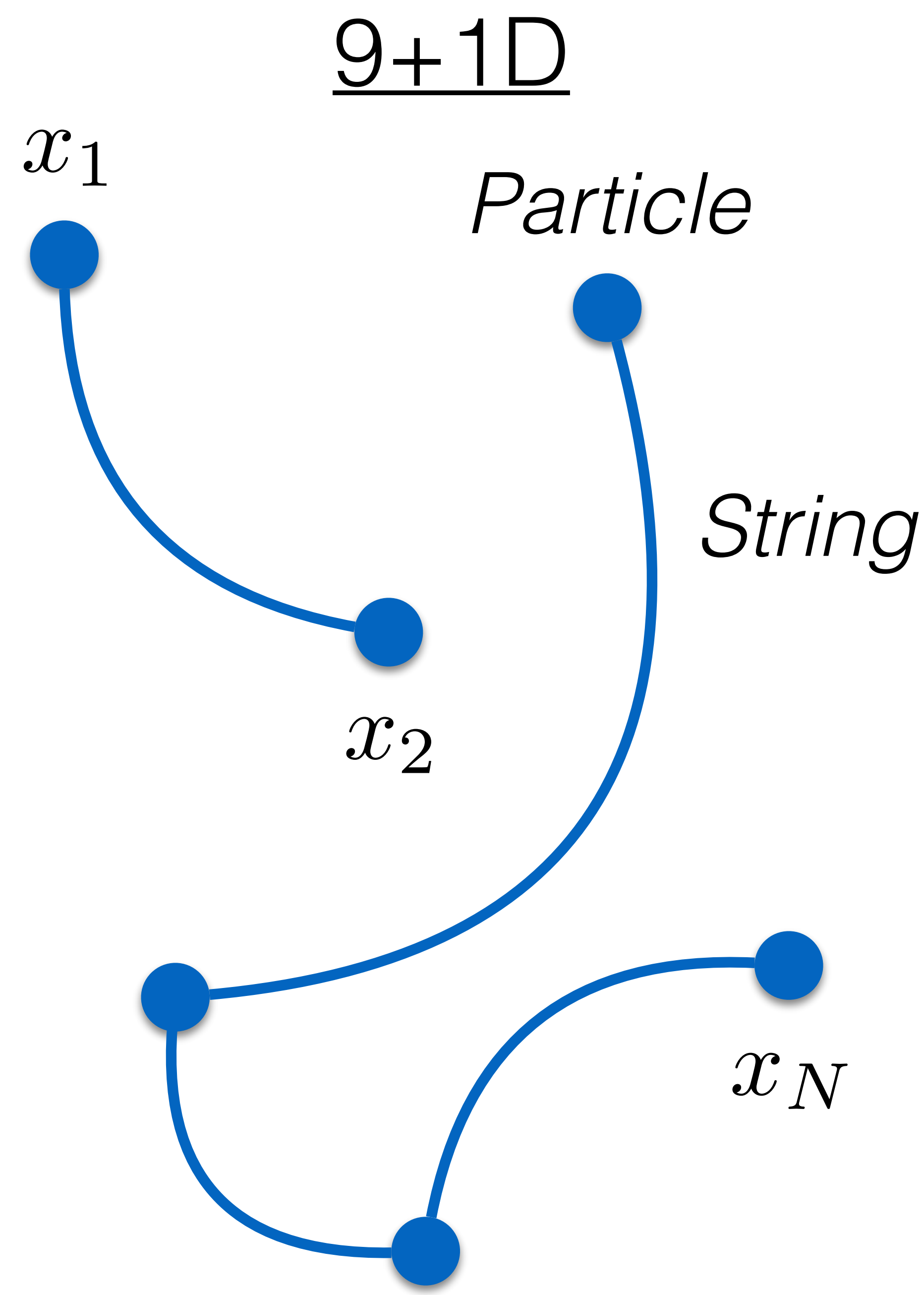
High temperature ⟷ Black hole

Large N,
strong coupling

Matrix gauge theory in 0+1D ⟷ Gravity
$X_1, \ldots, X_9$

[Banks, Fischler, Shenker, Susskind 1997]

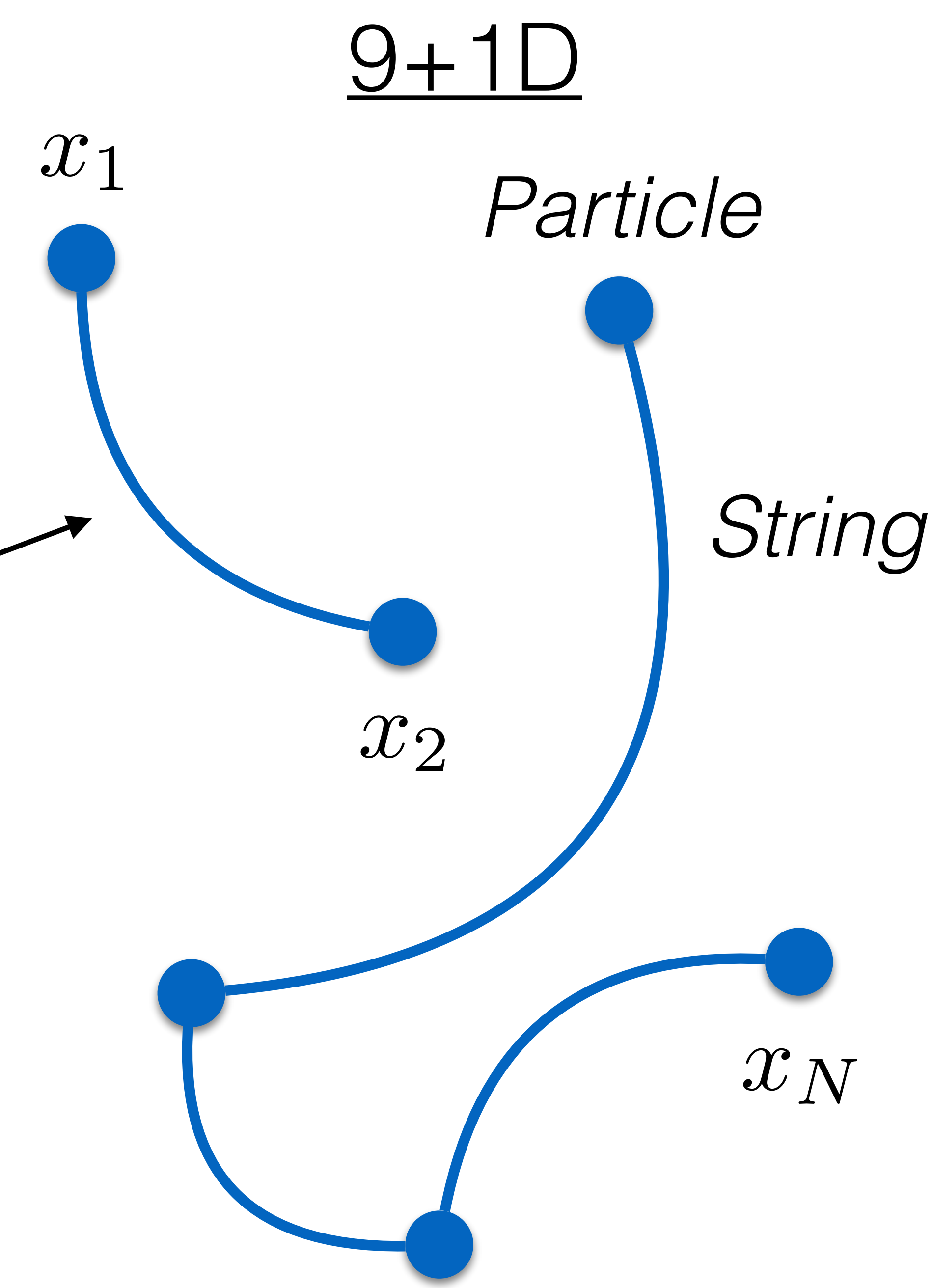
Matrix model holographic duality



Matrix model holographic duality

$$X_i = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{pmatrix}$$

$$X_i = \begin{pmatrix} x_1 & x_{12} & \\ x_{12}^* & x_2 & \\ & & \ddots \end{pmatrix}$$



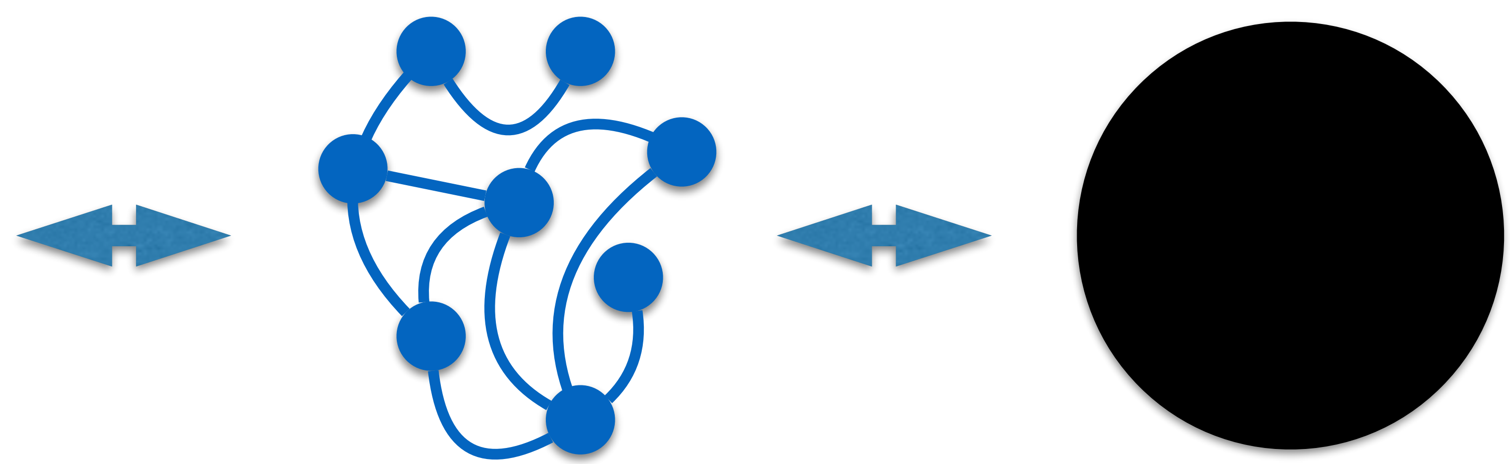
Holographic duality at finite temperature

Thermal state of matrix gauge theory

Black hole in a gravity theory

$$X_i = \left(\ \cdot\ \right)_{N\times N}$$

$$U \sim \mathrm{Tr}\,[X_i, X_j]^2$$



- Valid at large N, strong coupling
- We focus on weak coupling — large stringy corrections
- But: no phase transition in the coupling

Real-time dynamics
of matrix gauge theory

Weak coupling limit of matrix model

$$L = \frac{N}{2\lambda}\mathrm{Tr}\left(\sum_{i=1}^{9}(\dot{X}_i)^2 + \frac{1}{2}\sum_{i,j=1}^{9}[X_i, X_j]^2\right) + \cdots$$

- Effective dimensionless coupling: $\lambda_{\text{eff}} = \frac{\lambda}{T^3}$
- Large N, weak coupling / high temperature limit → **classical dynamics**
- Classical observables are functions of $\lambda_{\text{eff}}\, T^4$

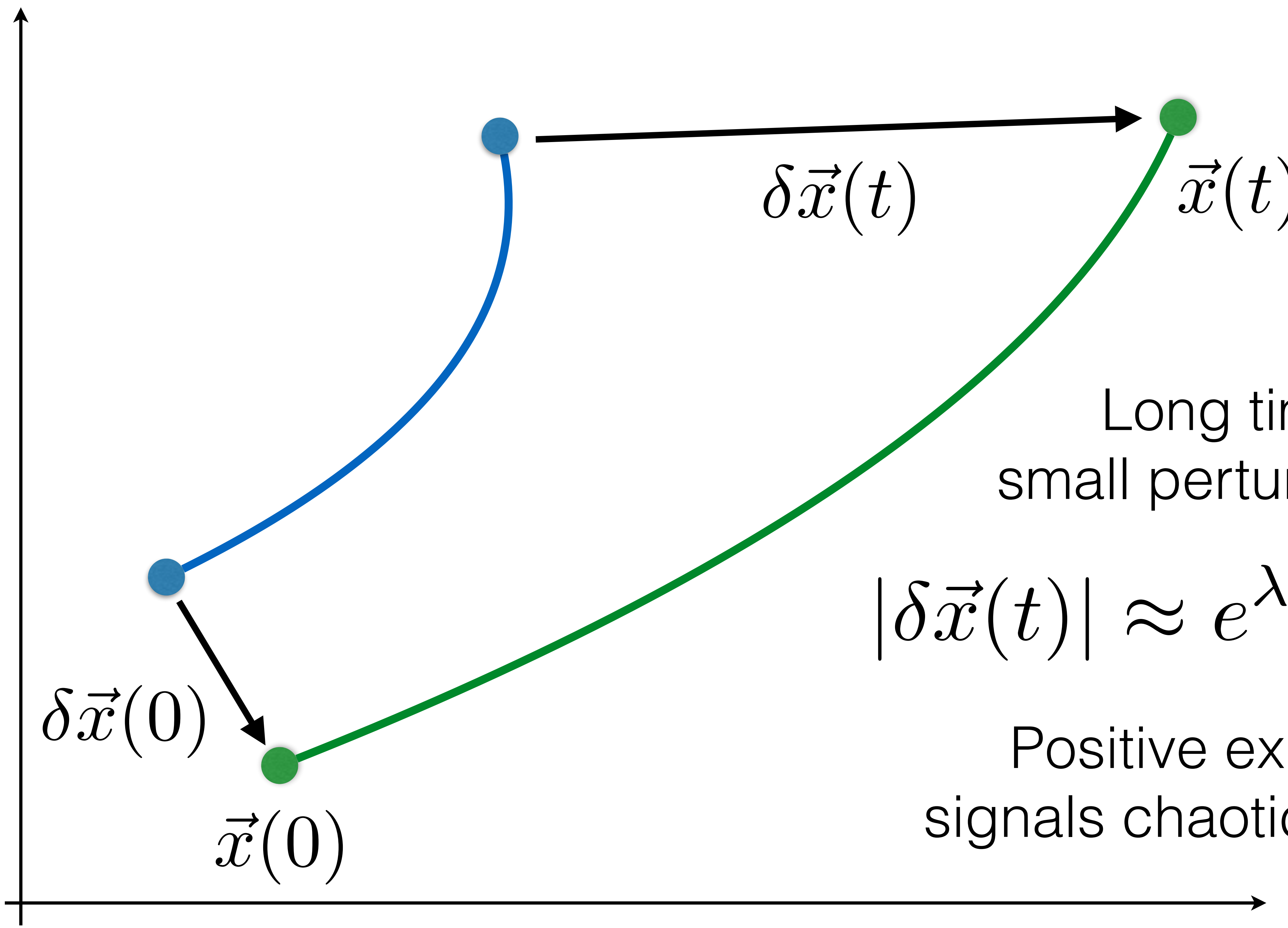
Equations of motion

$$L = \frac{N}{2\lambda}\mathrm{Tr}\left(\sum_{i=1}^{9}(\dot{X}_i)^2 + \frac{1}{2}\sum_{i,j=1}^{9}[X_i, X_j]^2\right) + \cdots$$

$$\begin{cases} \ddot{X}_i(t) = \sum_{j=1}^{9}[X_j, [X_i, X_j(t)]] \\ \sum_{i=1}^{9}[\dot{X}_i(t), X_i(t)] = 0 \end{cases}$$

Discretize and solve

Lyapunov exponent

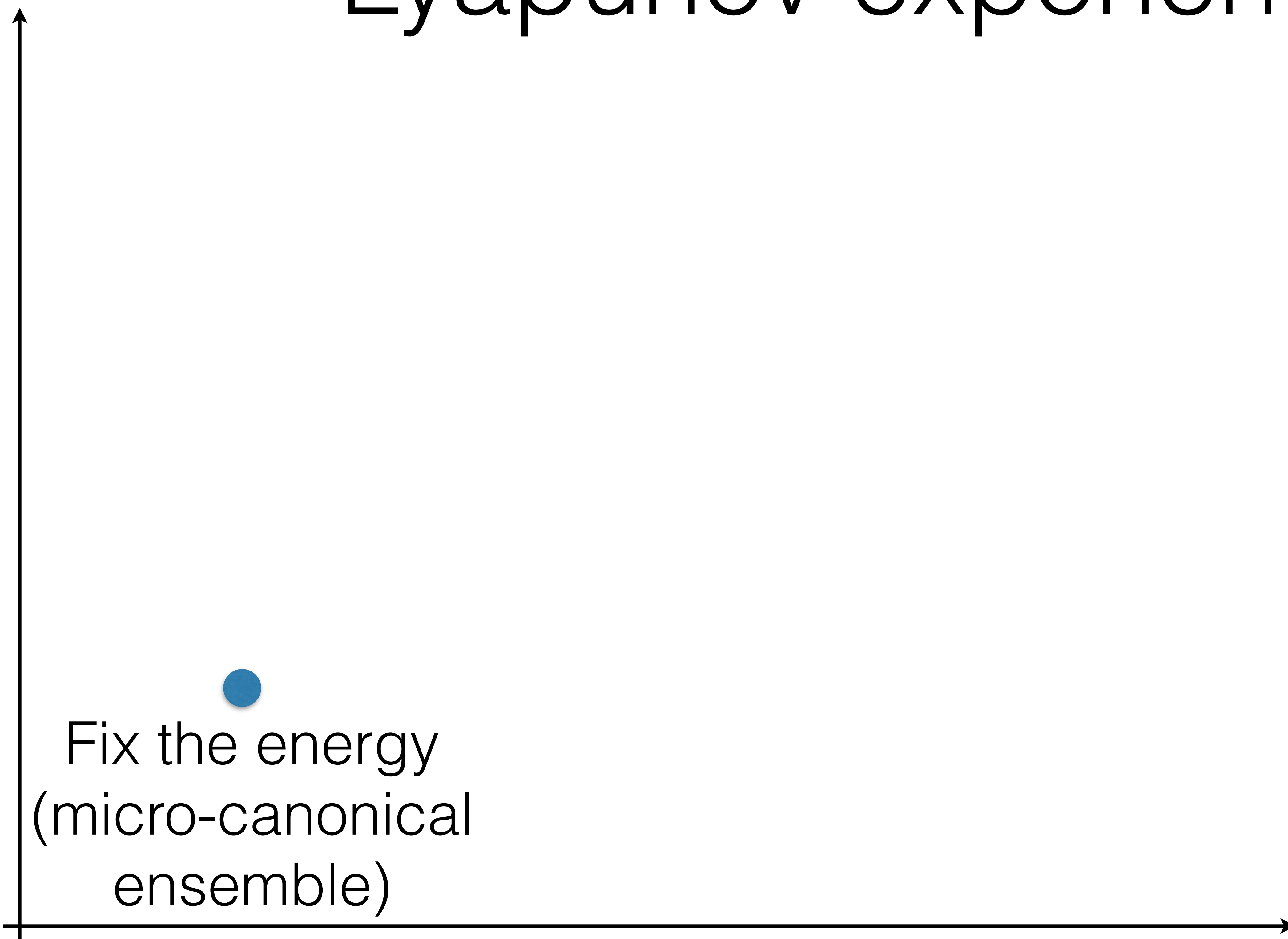


Long time,
small perturbation:

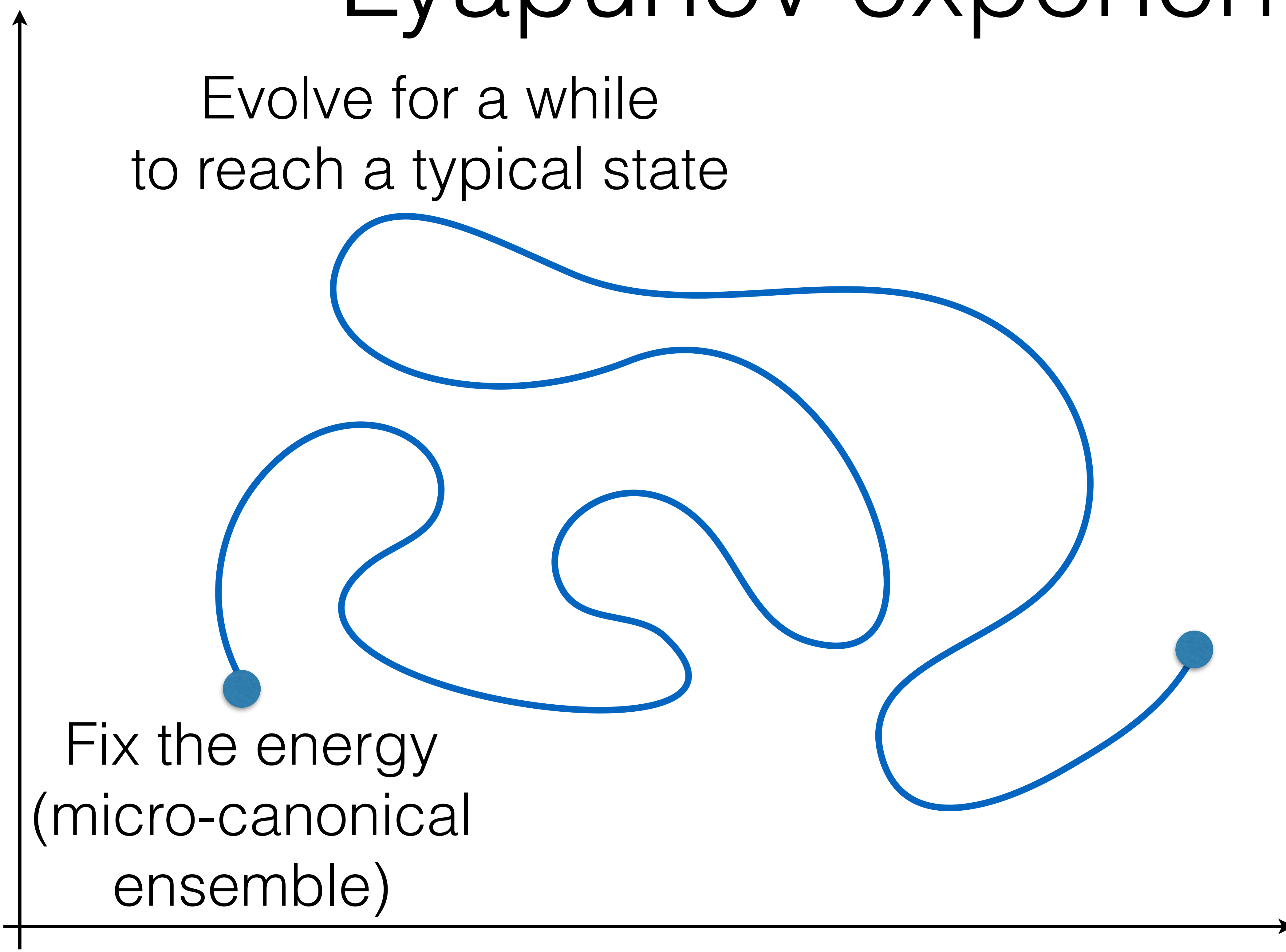
$$|\delta\vec{x}(t)| \approx e^{\lambda_L t}|\delta\vec{x}(0)|$$

Positive exponent
signals chaotic behavior

Measuring the Lyapunov exponent



Measuring the Lyapunov exponent



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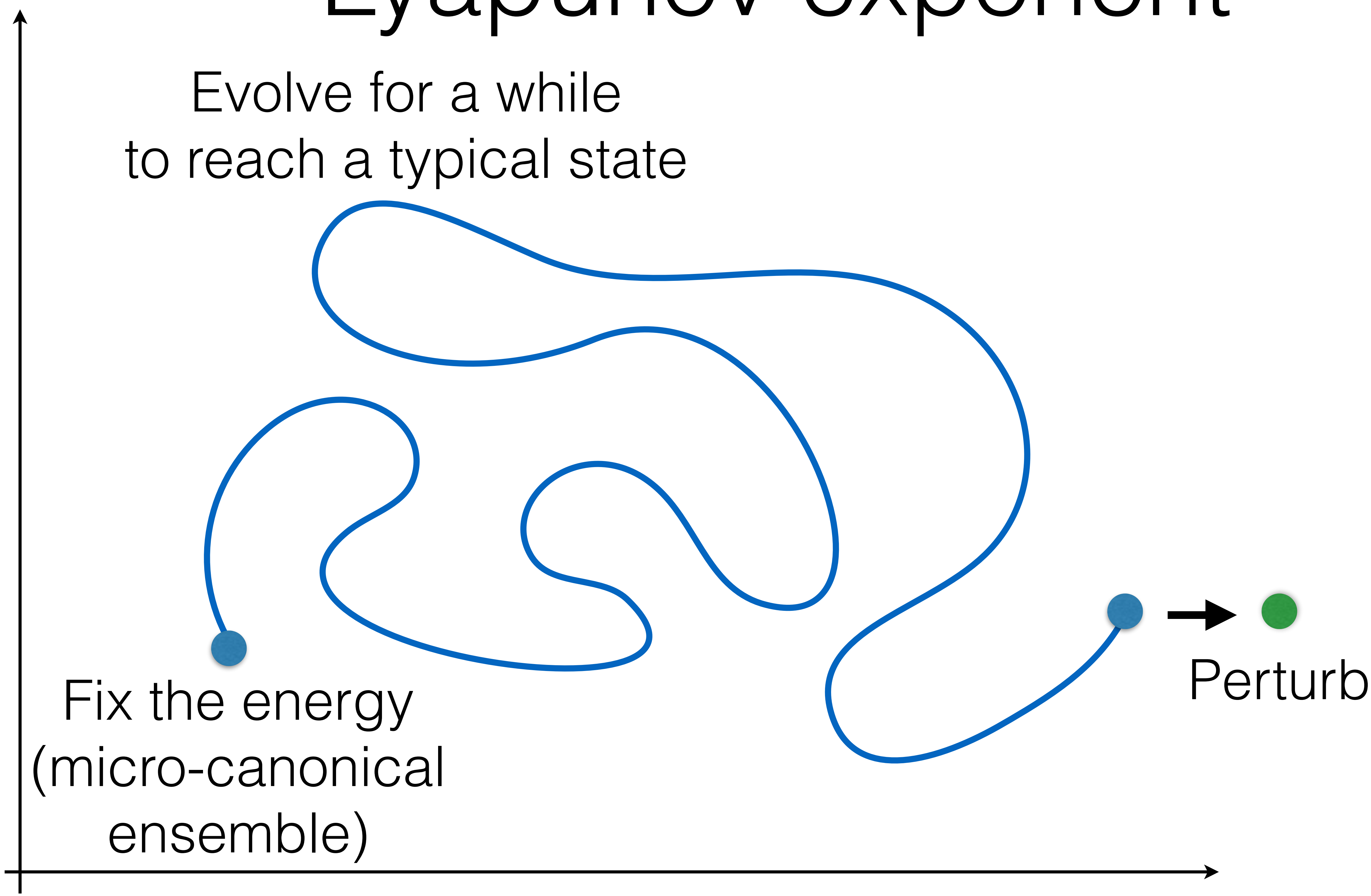
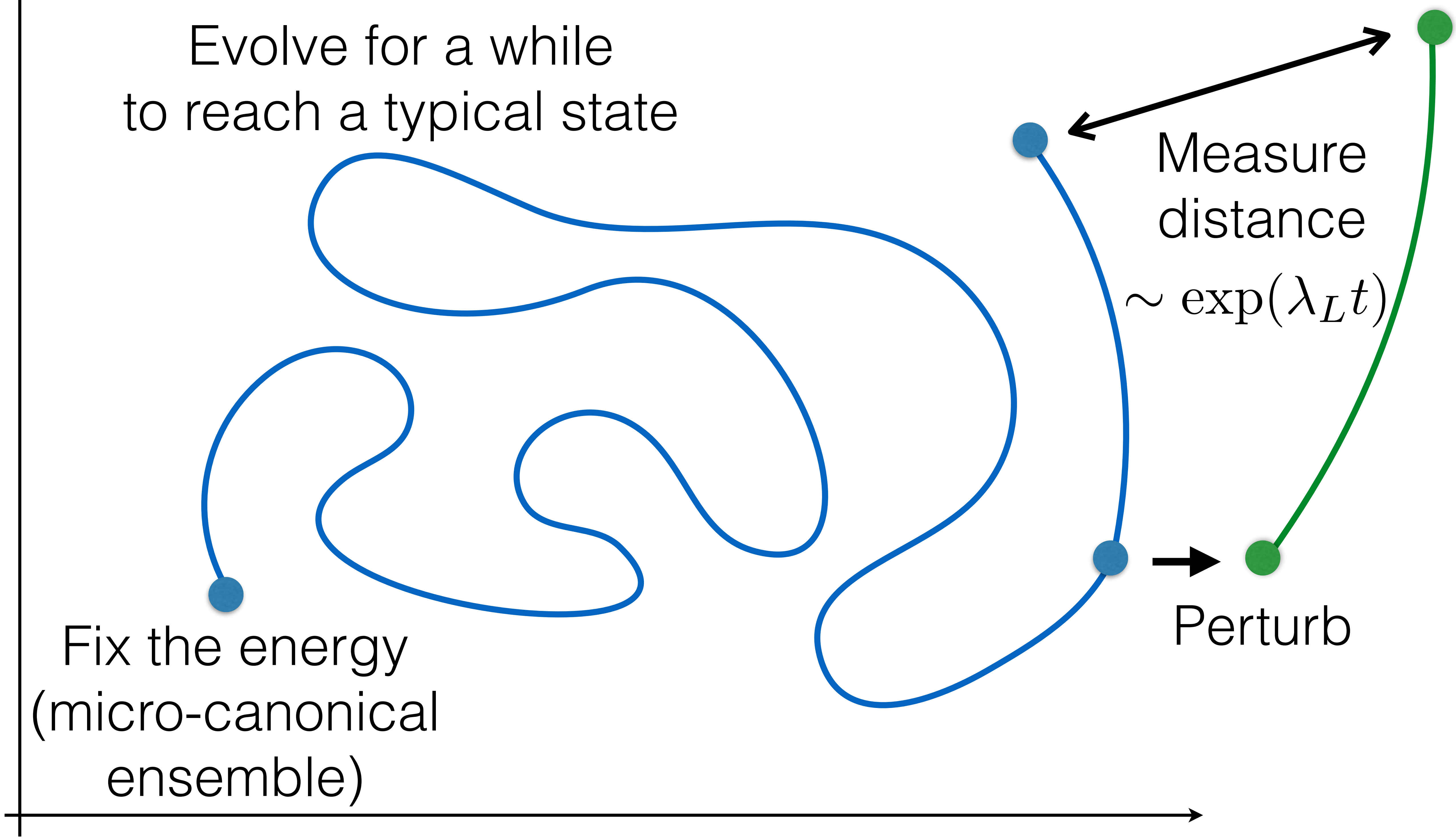
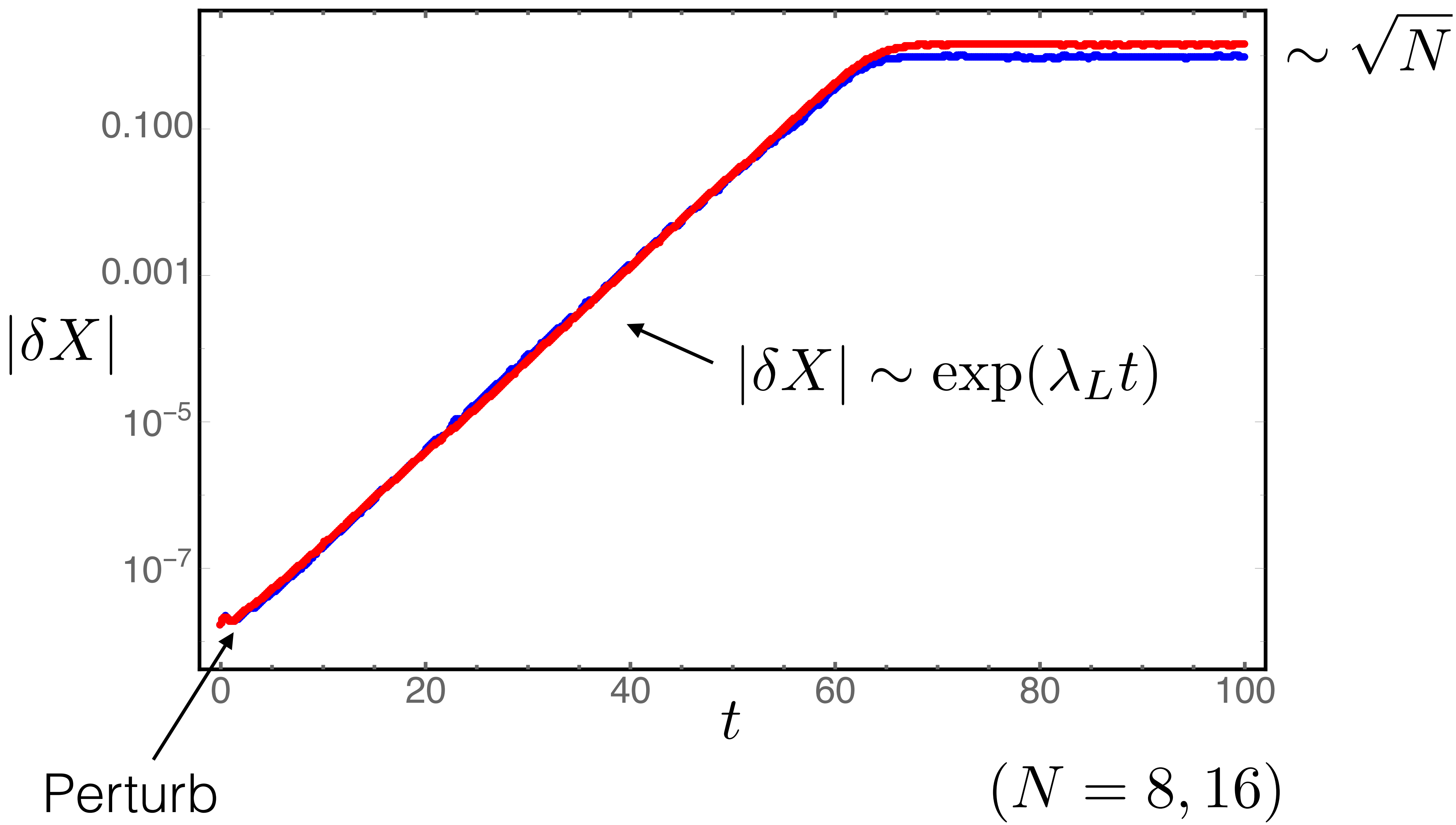


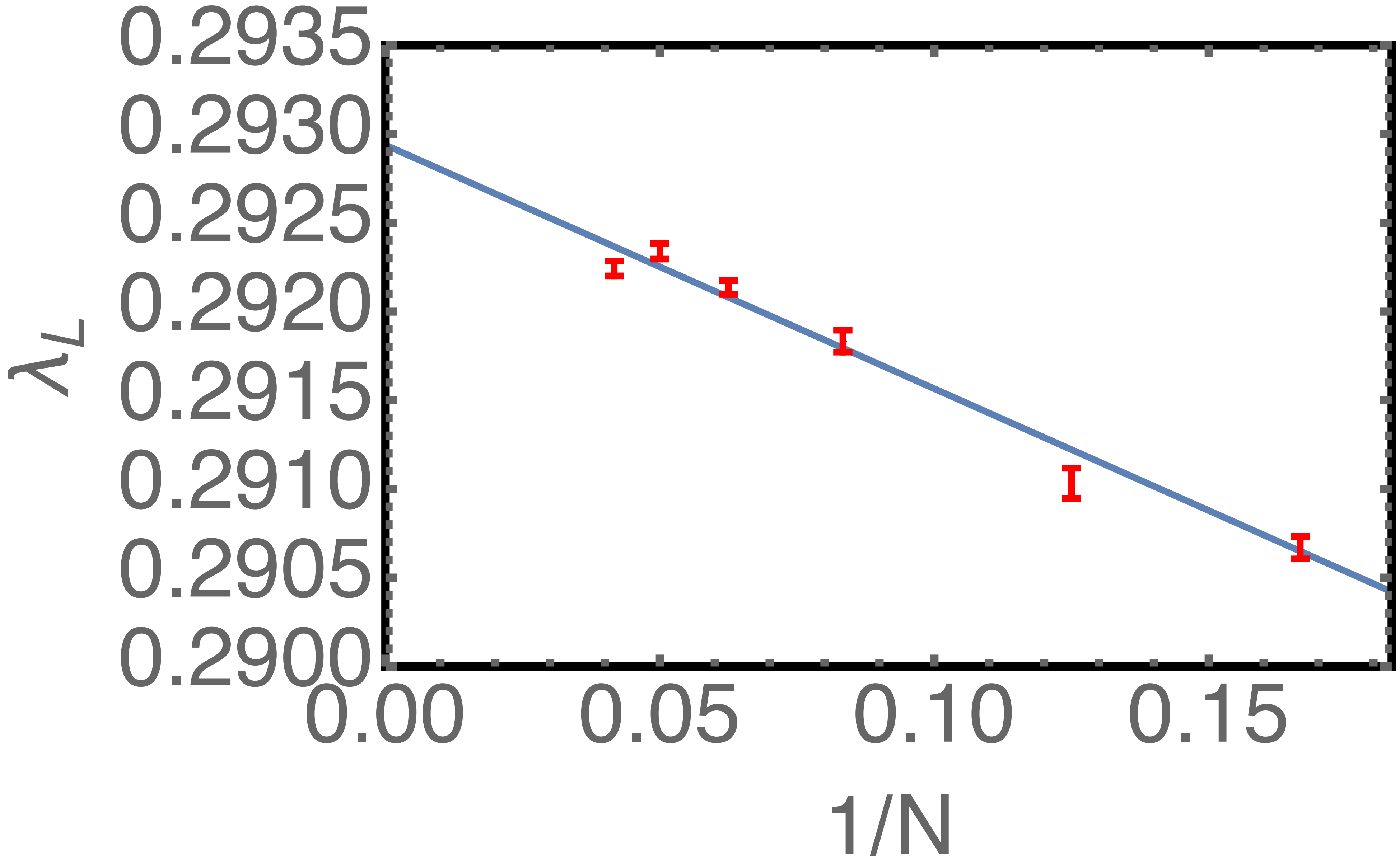
Measuring the Lyapunov exponent



Real-time exponential growth

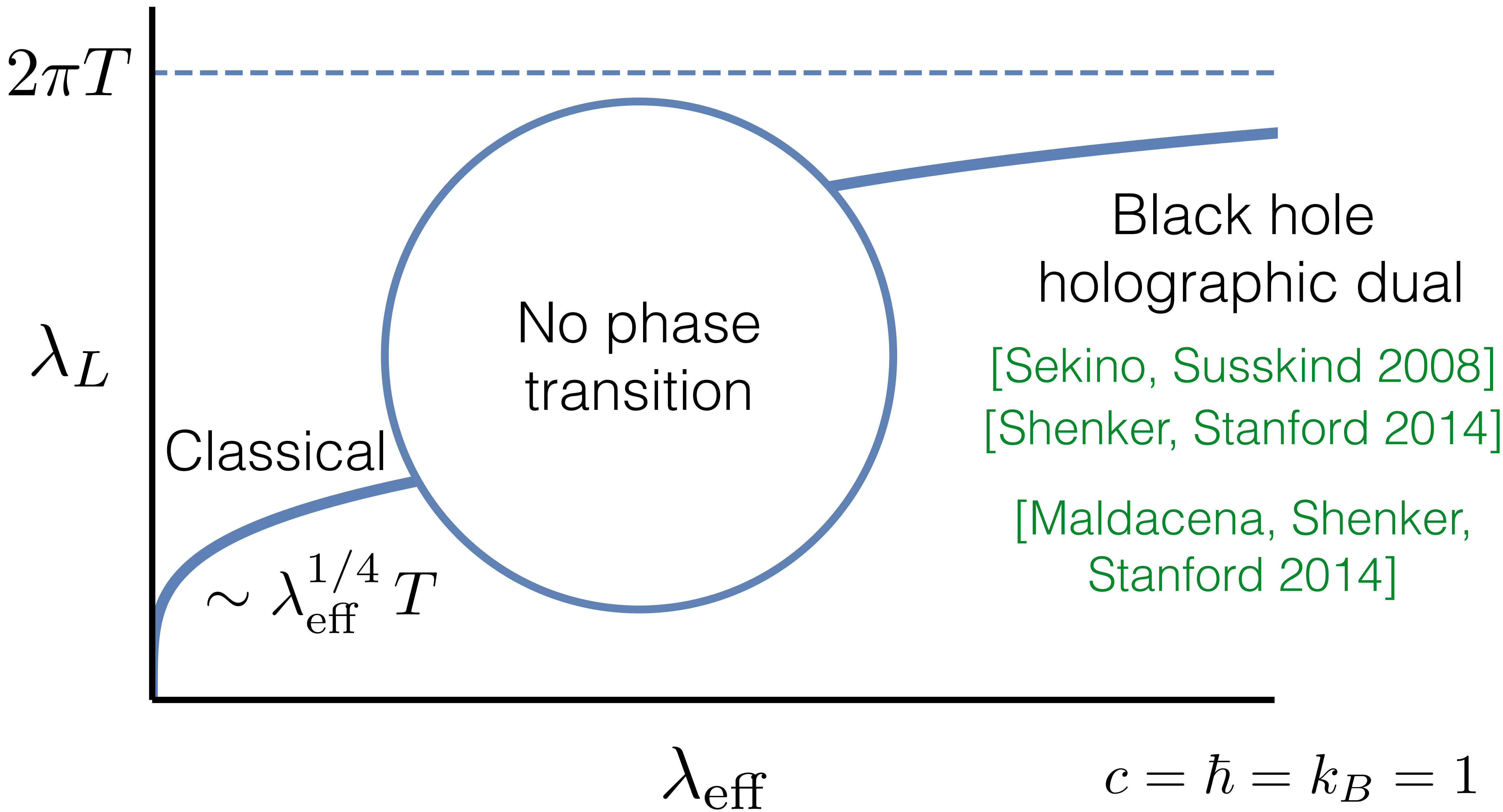


$1/N$ behavior



$$\lambda_L = \left[0.293 - \frac{0.014}{N} + O(1/N^2)\right]\lambda_{\text{eff}}^{1/4}\,T \qquad (N = 6, \ldots, 24)$$

Lyapunov exponent: gauge theory vs. black holes

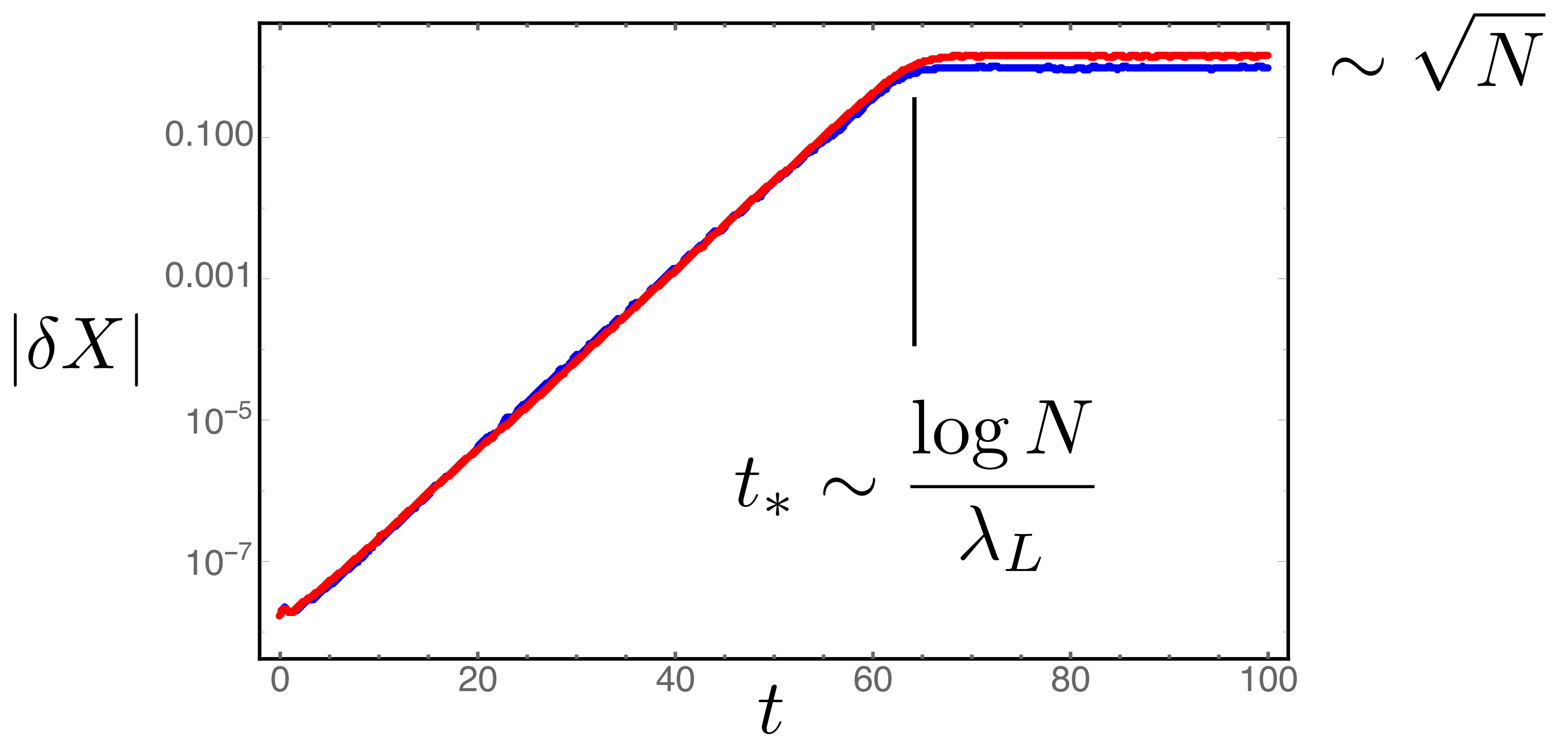


$c = \hbar = k_B = 1$

Scrambling time and large N behavior

- Scrambling time =
  time to completely de-localize a local perturbation

- Numerically: $\exp(\lambda_L t_*) \sim \sqrt{N}$



Scrambling time and large N behavior

- Scrambling time = time to completely de-localize a local perturbation

- Black holes are 'fast scramblers':

$$t_* \sim \log S \sim \log N$$

- Same behavior in our system:

$$\lambda_L \sim N^0 , \qquad t_* \sim \frac{\log N}{\lambda_L} \sim \log N$$

Real-Time Correlators

Lyapunov exponent from correlators

$$\langle O(0)O(t)\rangle = \frac{1}{T}\int_0^T dt'\, O(t')O(t'+t)$$

$$\langle O(0)O(t)\rangle - \langle O\rangle^2 \sim \exp(-\tilde{\lambda}t)$$

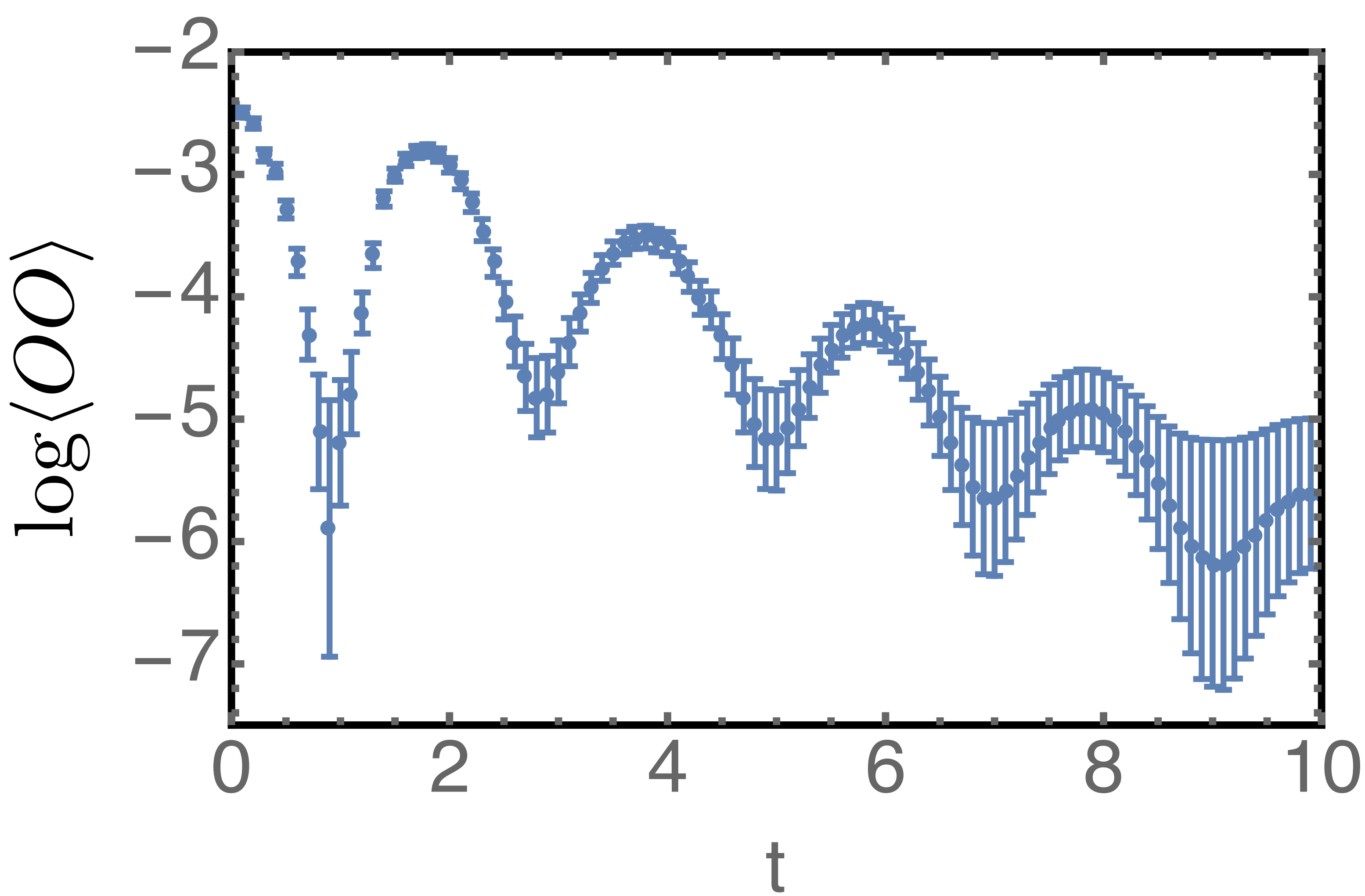
$$\tilde{\lambda} \approx \lambda_L\, ??$$

Motivation: Making contact with the quantum theory

Lyapunov exponent from correlators

- Choose operators with vanishing 1-point functions

$$\left\langle \mathrm{Tr}(X_i X_j)(0)\ \mathrm{Tr}(X_i X_j)(t) \right\rangle \qquad (i \neq j)$$



Lyapunov exponent from correlators

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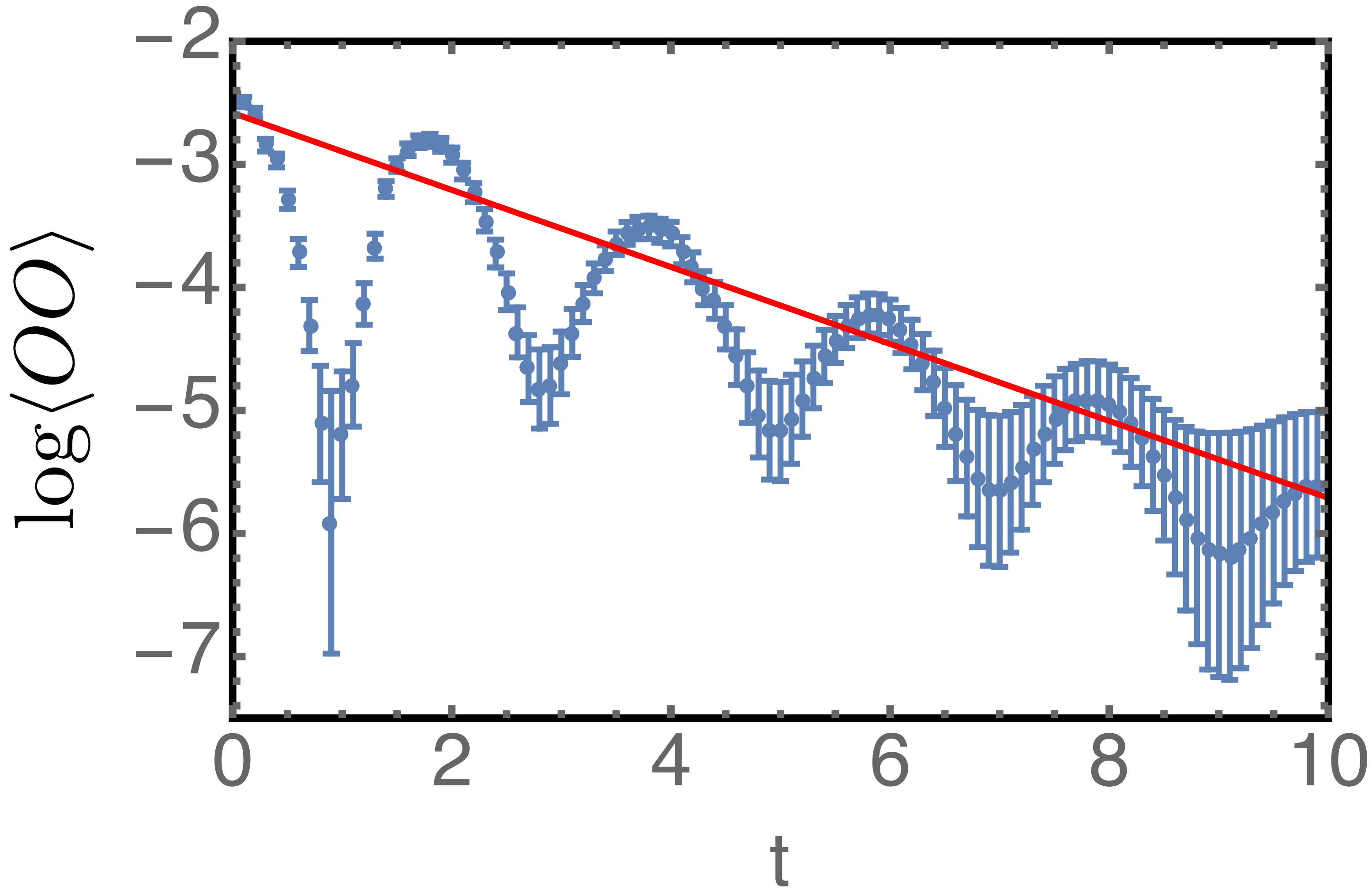
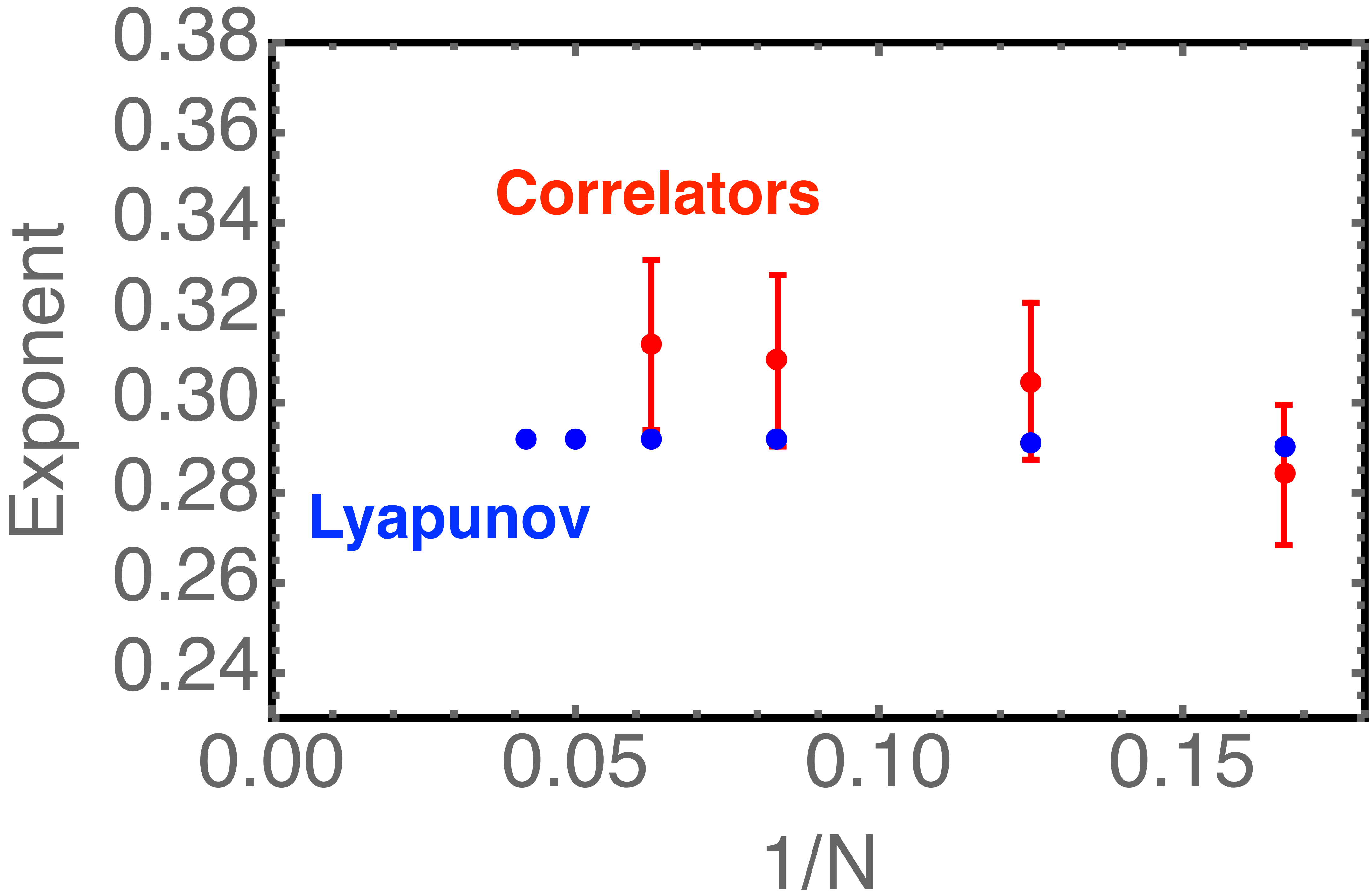


Lyapunov exponent vs. correlator exponents



Lyapunov Spectrum

Lyapunov spectrum

- Generic perturbation grows as:

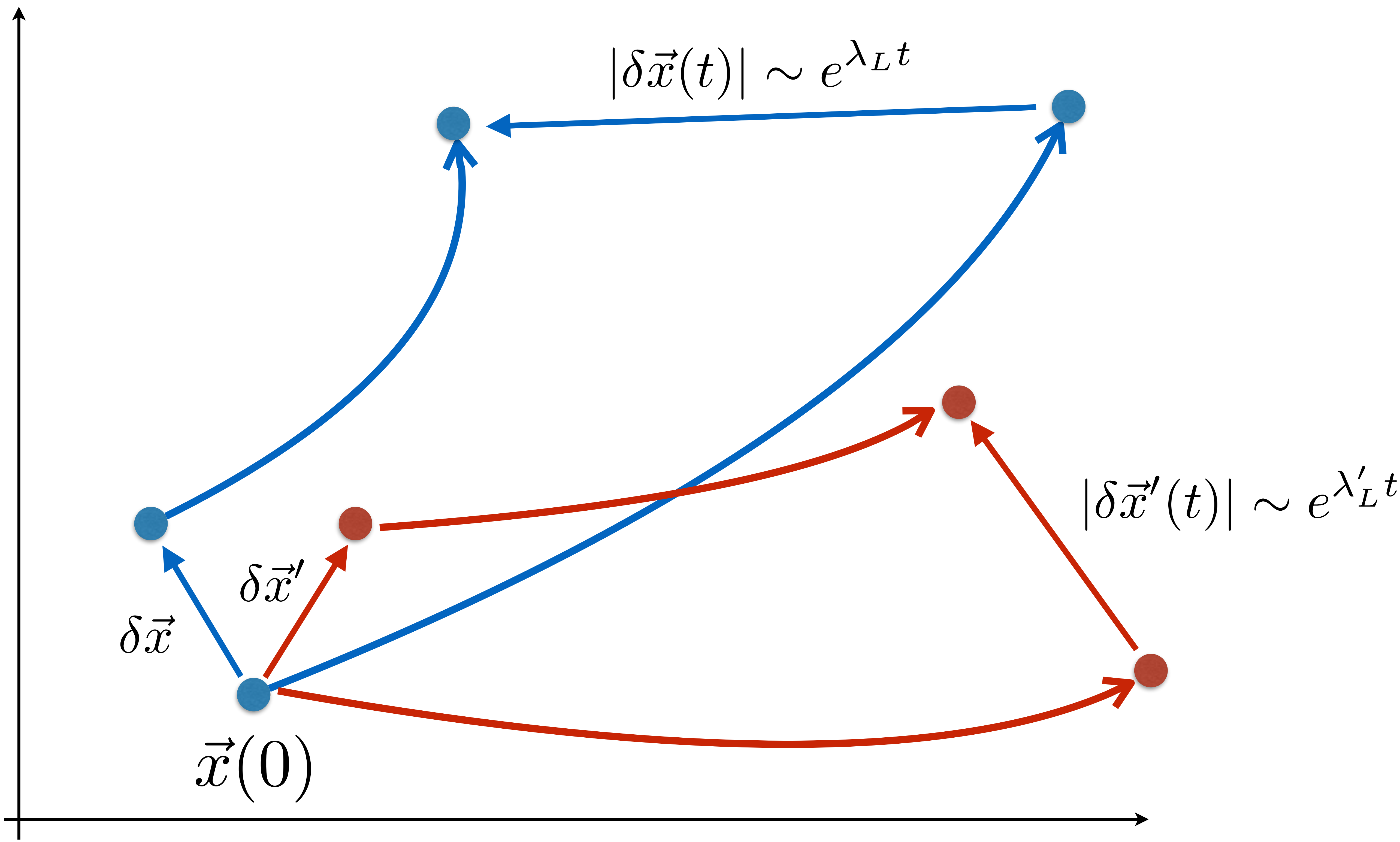
$$|\delta\vec{x}(t)| \approx \exp(\lambda_L t)|\delta\vec{x}(0)|$$

- Tuned perturbations have different exponents
- There is a spectrum of Lyapunov exponents:

no. exponents = dimension of phase space

- Lyapunov exponent λ_L is the largest eigenvalue

Lyapunov spectrum



Lyapunov spectrum motivation

- Detailed information about chaotic behavior
- Kolmogorov-Sinai entropy measures rate of entropy production
  - Approximated by sum of positive exponents

[Kunihiro, Müller, et al. 2010]

Measuring the spectrum

- Lyapunov spectrum = spectrum of transfer matrix

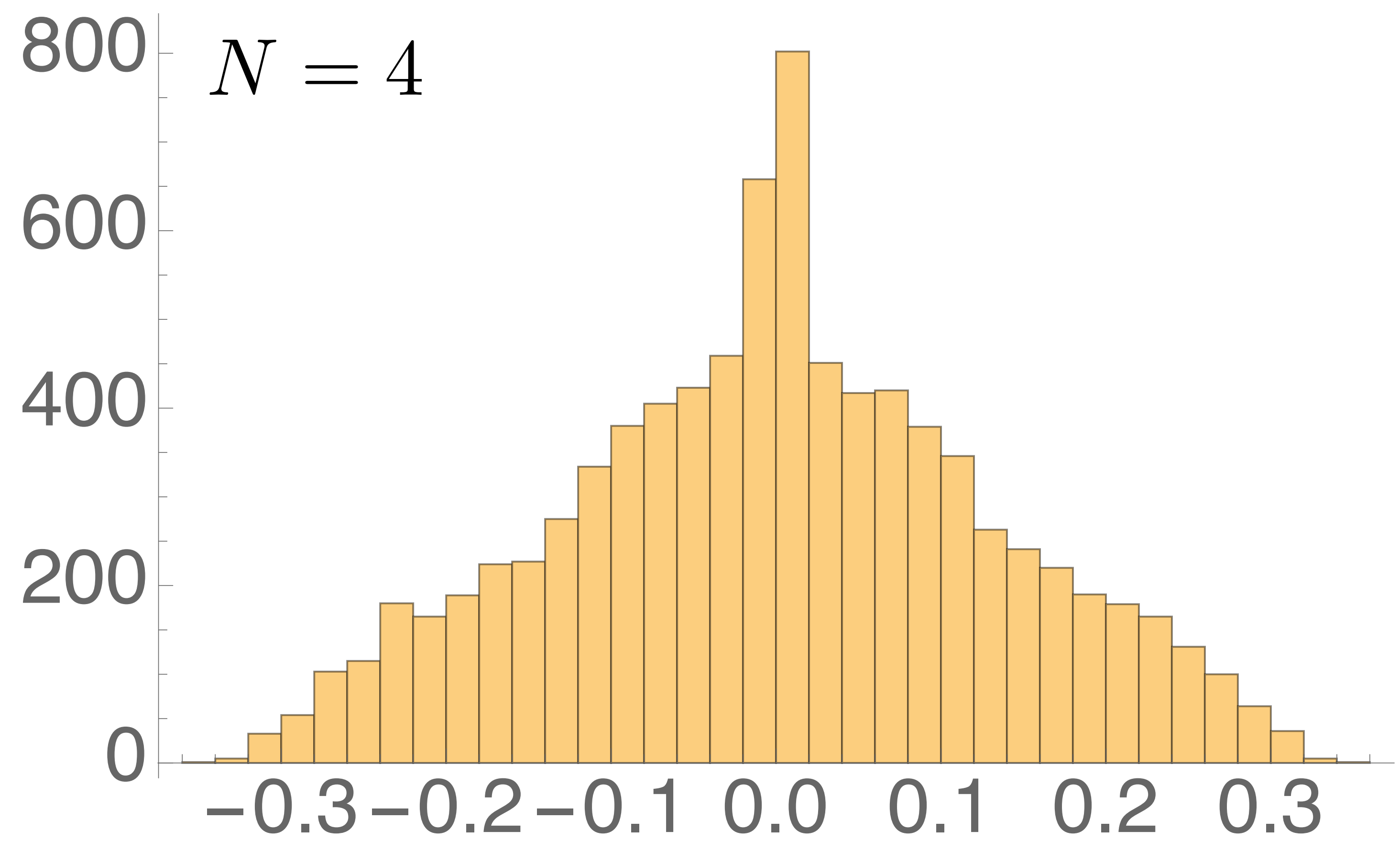
$$\delta \dot{X}_i = \delta V_i \, , \quad \delta \dot{V}_i = [\delta X_j, [X_i, X_j]] + \cdots$$

$$\begin{pmatrix} \delta \dot{X} \\ \delta \dot{V} \end{pmatrix} = U(t) \begin{pmatrix} \delta X \\ \delta V \end{pmatrix}$$

Global Lyapunov spectrum

- Measure spectrum of *U(t)* at late times

$$\begin{pmatrix} \delta \dot{X} \\ \delta \dot{V} \end{pmatrix} = U(t) \begin{pmatrix} \delta X \\ \delta V \end{pmatrix}$$



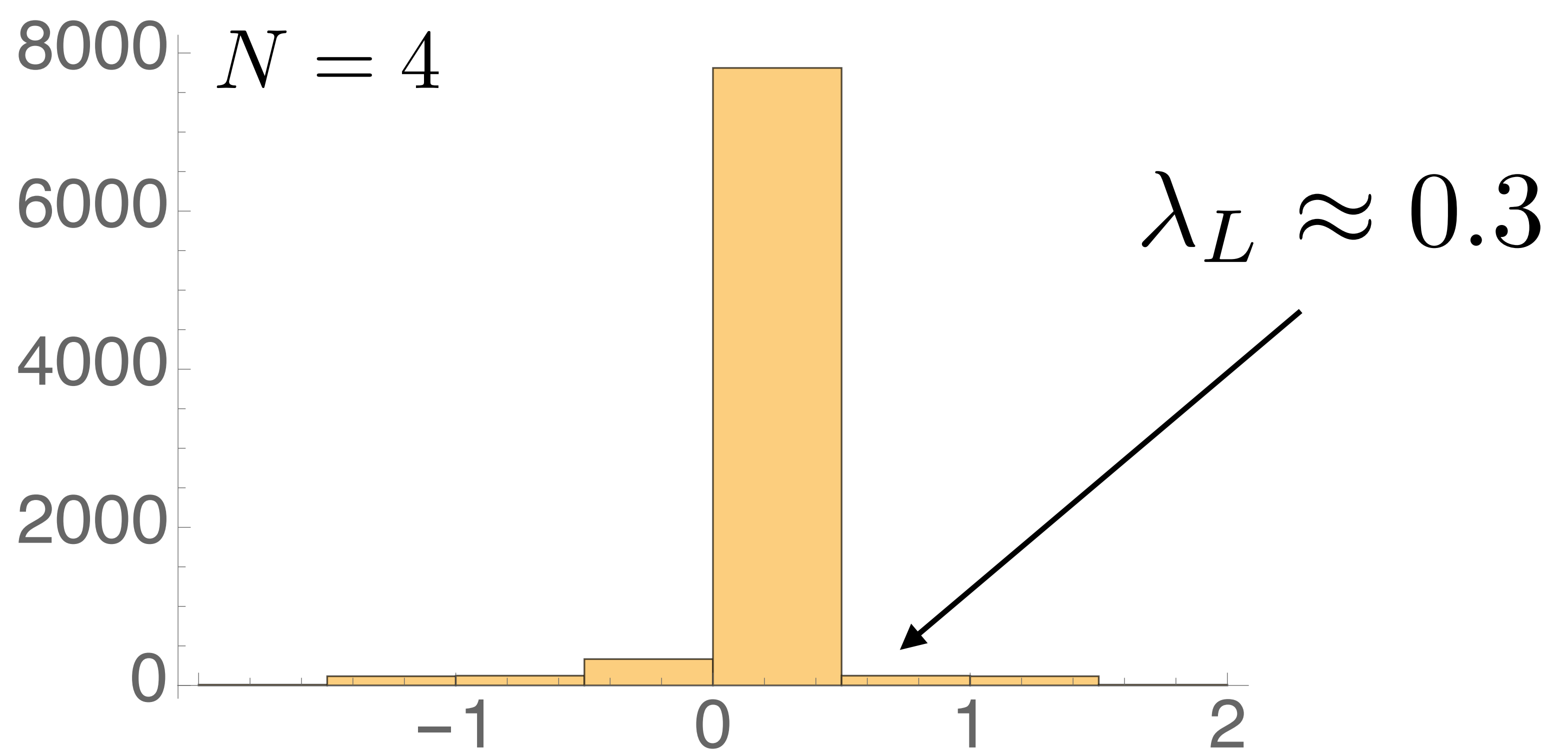
Partial Results

$$\lambda_L \approx 0.3$$

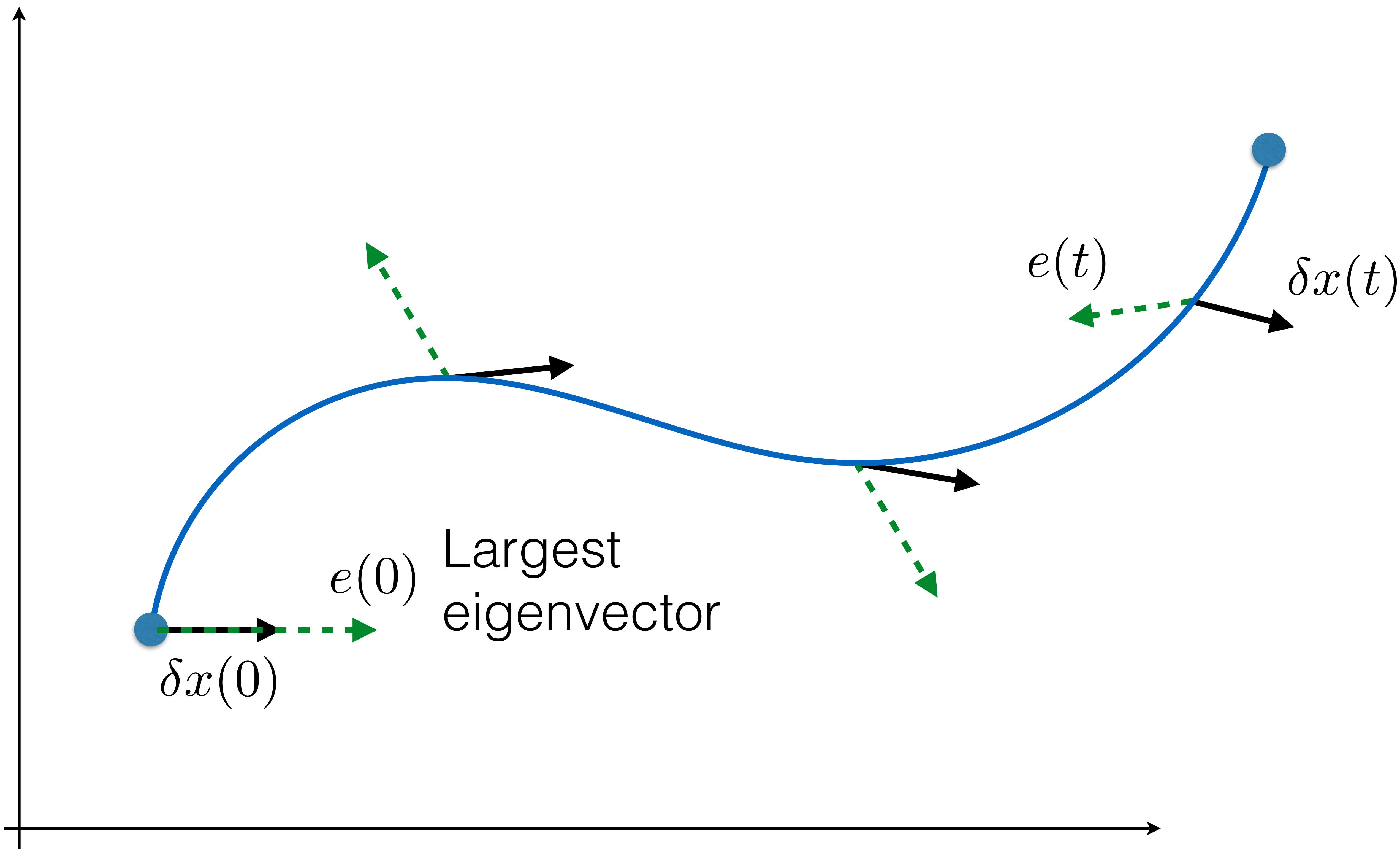
Local Lyapunov spectrum

- Spectrum of *U(t)* at short time
- Lyapunov exponent is not the largest eigenvalue

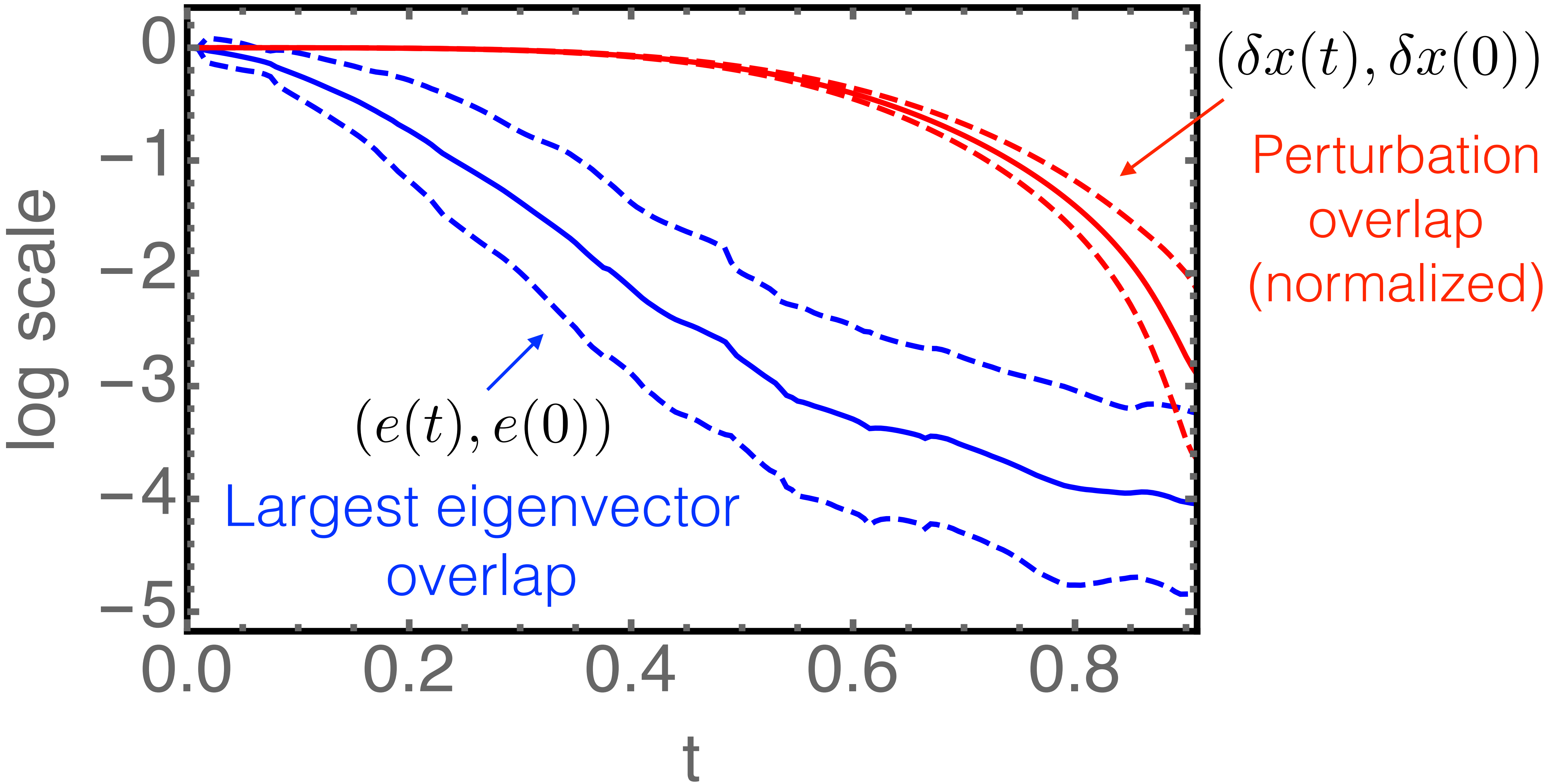
Partial Results



Fast eigenvector evolution



Fast eigenvector evolution



Perturbation cannot catch up with evolving eigenvector!

Random matrix toy model

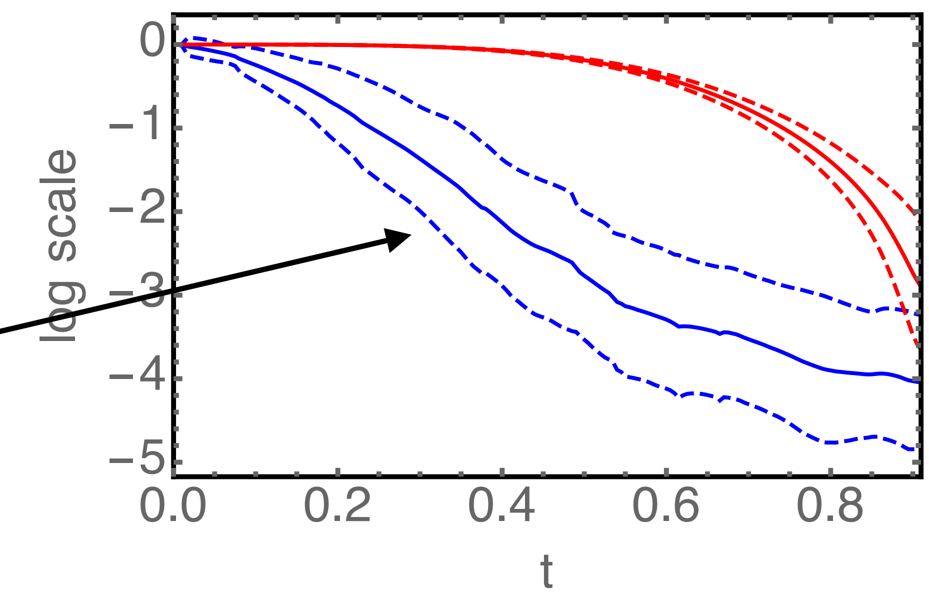
- Toy model: Evolve with random matrices

$$\delta x(t) = (U_n)^{dt} \cdots (U_1)^{dt} \delta x(0) \sim e^{\lambda t} \delta x(0)$$

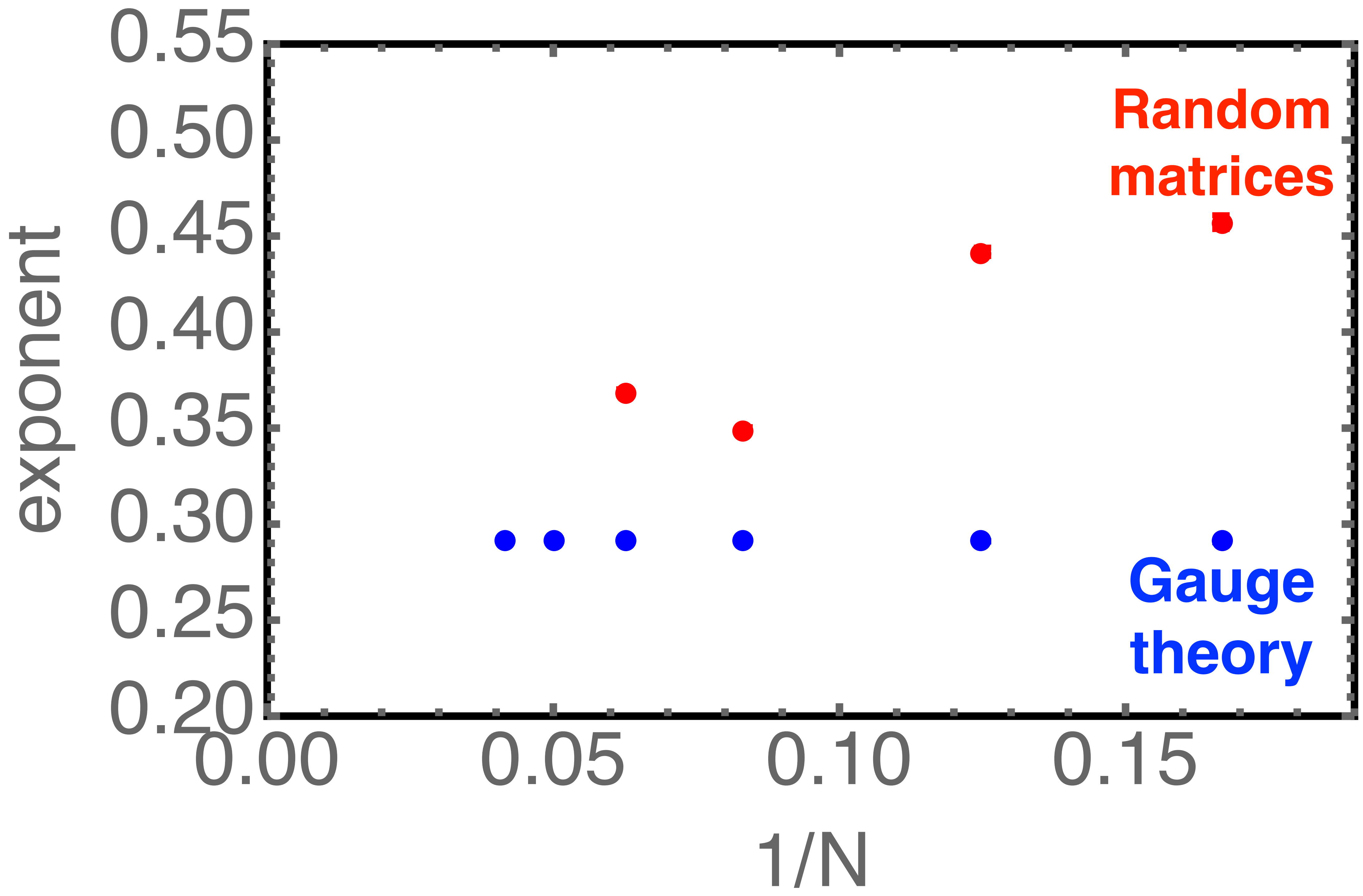
- Spectrum + time step taken from gauge theory

- Measure exponent

dt is
typical decay time

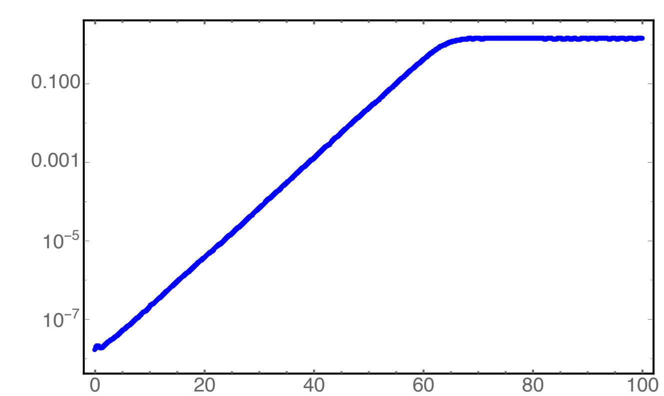


Random matrix exponents



Summary

- Real-time chaotic behavior in classical gauge theory
- Lyapunov exponent converges at large N
- Fast scrambling behavior, similar to black holes
- Relation to real-time correlators
- Going to higher dimensions:
  - UV cascade, classical approx. may break down



Thank You!

Discretization

$$V_i(t) = \dot{X}(t)$$

$$F_i(t) = \sum_{j=1}^{9} [X_j(t), [X_i(t), X_j(t)]]$$

$$X_i(t+dt) = X_i(t) + V_i(t)dt + F_i(t)\frac{dt^2}{2}$$

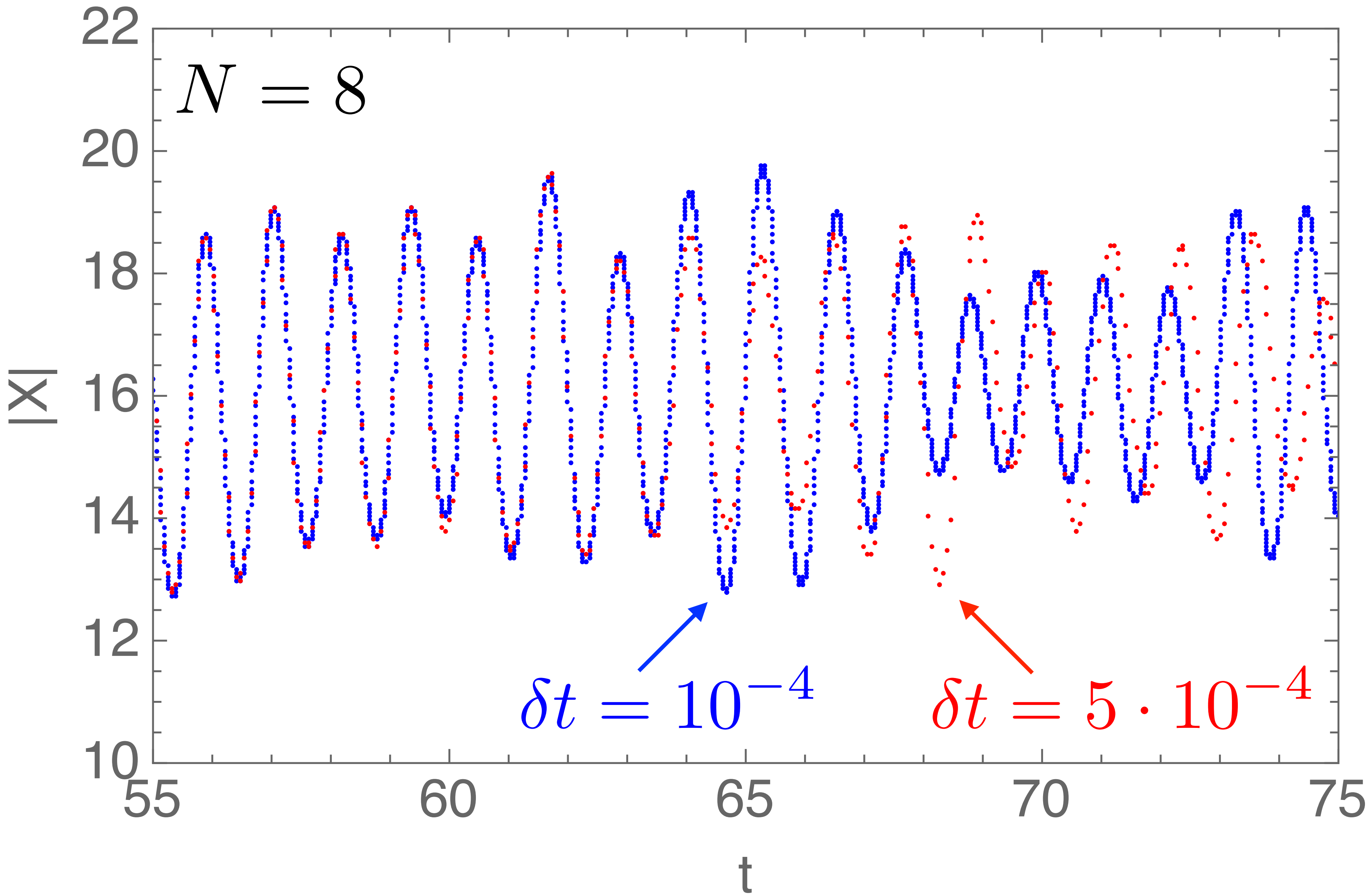
$$V_i(t+dt) = V_i(t) + [F_i(t) + F_i(t+dt)]\,\frac{dt}{2}$$

Sprott's algorithm

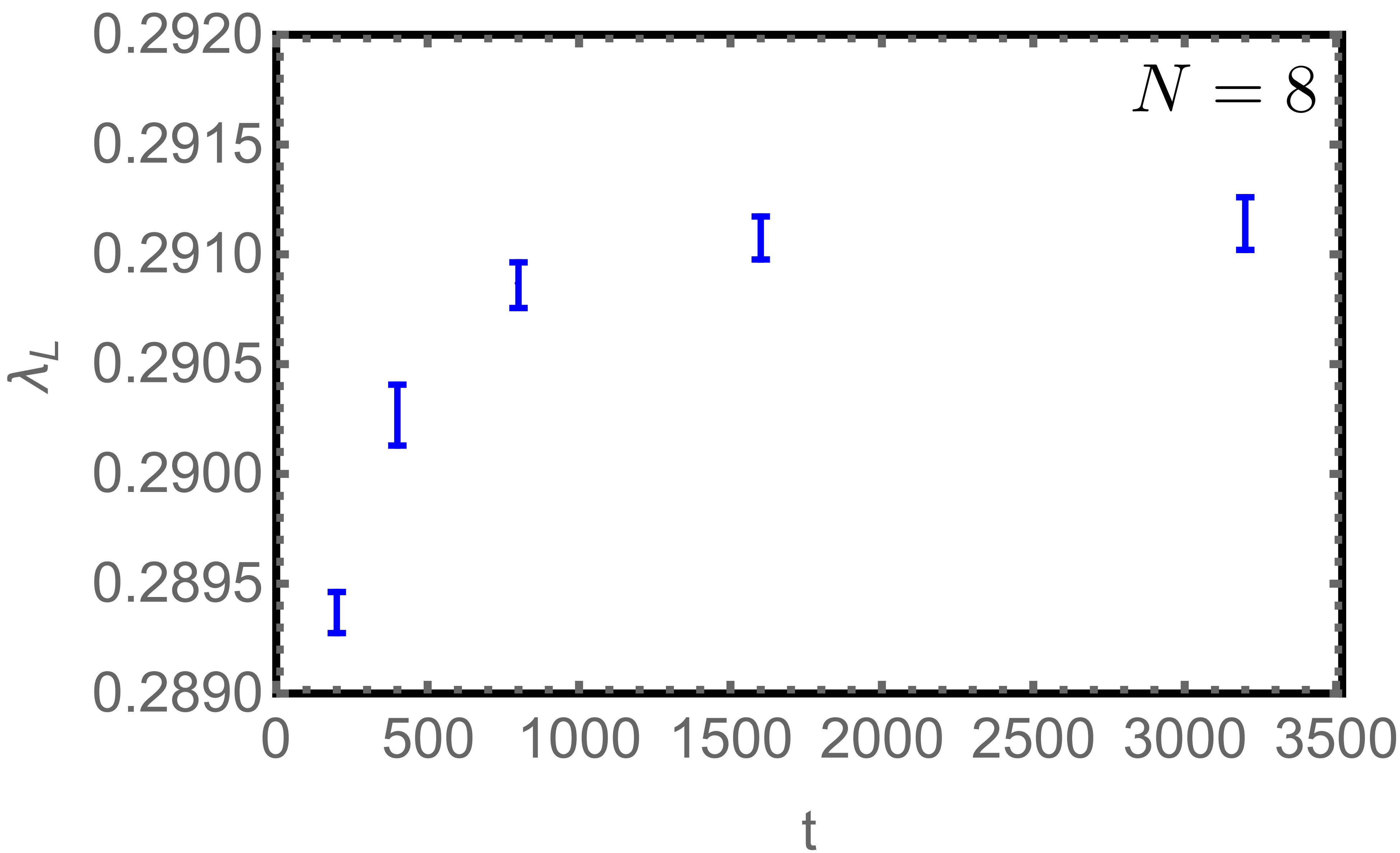
- Make a small perturbation *X -> X+dX*
- Evolve one time step *dX -> dX'*
- Record log(|dX'| / |dX|)
- Rescale |dX'| -> |dX|

$$\left\langle \log \frac{|\delta X'|}{|\delta X|} \right\rangle \to \lambda_L$$

Error accumulation



Exponent vs. thermalization time



Classical coupling dependence

$$\lambda E = \frac{N}{2}\mathrm{Tr}\left(\dot{X}^i + [X^i, X^j]^2\right)$$

$$\lambda_L = f(\lambda T) = f(\lambda_{\mathrm{eff}} T^4) \sim T$$

$$\lambda_L \sim \lambda_{\mathrm{eff}}^{1/4}\, T$$