TEST OF QFEM*:
2D ISING CFT ON RIEMANN SPHERE

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*Quantum Finite Element Method
HISTORY: WHERE AM I?

(5+ YEARS AFTER BIRTH OF LATTICE QCD)

Courtesy Special Collections, UC Santa Cruz
20+ Years to adapt RG for Multigrid Lattice Dirac Operators

Combining renormalization group and multigrid methods (Brower, Giles, Moriarty, Tamayo)


ADAPTIVE SMOOTH AGGREGATION ALGEBRAIC MULTIGRID

32³ x 256 aniso clover on 1024 BG/P cores
mixed precision BiCGStab
mixed precision multigrid (old)
mixed precision multigrid (new)

seconds per solve

mass

m_{phys} m_{light} m_s
OUTLINE

Motivation: Conformal Fixed Point
Curved Manifolds via FEM + Regge (IR)
Counter Terms for Multiple UV cut-off.
Test Case: $c = 1/2$ 2D CFT (e.g. Ising)
Future Tasks: 3D, Fermions, Gauge Theory etc.
Motivation for FEM Lattice Field Theory

- Conformal Field Theories, interesting for
  - BSM composite Higgs
  - AdS/CFT weak-strong duality
  - Model building & Critical Phenomena in general

Potential Huge Advantage for CFT!

- Linear Hypercubic $\quad$ vs $\quad$ Exponential Radial Lattice

$$a < r < aL \quad\rightarrow\quad 1 < \log(r) < L$$

Both UV asy freedom and IR conformal on a lattice?
Radial Quantization

\[ \mathbb{R}^d \quad \mapsto \quad \mathbb{R} \times \mathbb{S}^{d-1} \]

Evolution:

\[ H = P_0 \text{ in } t \quad \mapsto \quad D \text{ in } \tau = \log(r) \]

\[ ds^2 = dx^\mu dx_\mu = e^{2\tau} \left[ d\tau^2 + d\Omega^2 \right] \]

"time" \hspace{1em} \tau = \log(r) \hspace{1em} "mass" \hspace{1em} \Delta = d/2 - 1 + \eta

\[ D \rightarrow x_\mu \partial_\mu = r \partial_r = \frac{\partial}{\partial \tau} \]
EXACT CFT: POWER LAW CORRELATOR

Conformal correlator:  \[ \langle \phi(x_1)\phi(x_2) \rangle = C \frac{1}{|x_1 - x_2|^{2\Delta}} \]

\[ r_1^\Delta r_2^\Delta \langle \phi(\tau_1, \Omega_1)\phi(\tau_2, \Omega_2) \rangle = C \frac{1}{[r_2/r_1 + r_1/r_2 - 2 \cos(\theta_{12})]^\Delta} \]

\[ \geq C e^{-(\log(r_2) - \log(r_1))\Delta} \]

\[ = C e^{-\tau\Delta} \]

With \[ |x_1 - x_2|^2 = r_1 r_2 [r_2/r_1 + r_1/r_2 - 2 \cos(\theta_{12})] \]

as \[ \tau = \log(r_2) - \log(r_1) \rightarrow \infty \]
(CFTS: NO LOCAL LAGRANGIAN)

(i.e. Data: spectra + couplings to conformal blocks)

\[ \langle \phi(x_1)\phi(x_2) \rangle = C \frac{1}{|x_1 - x_2|^{2\Delta}} \]

\[ \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}} \mathcal{O}_k(0) \]

Only “tree” diagrams! “partial waves” exp: sum over conformal blocks

CFT Bootstrap: OPE & factorization completely fixed the theory
INEQUALITIES FROM BOOTSTRAP*

**FIRST TOY PROBLEM: 3-D ISING AT WILSON-FISHER**

\[ Z_{\text{Ising}} = \sum_{\sigma(x,t) = \pm 1} e^{\beta \sum_{t,\langle x, y \rangle} \sigma(t,x)\sigma(t,y) + \beta \sum_{t,x} \sigma(t + 1, x)\sigma(t, x)} \]

\[ \tau = \log \rho \]

\((r = 0) \quad +\infty \quad -\infty\)
ORDER S Refined Triangulated Icosahedron

$s = 1$

$s = 8$

$l = 0 (A), l (T1), 2 (H)$ are irreducible 120 Icosahedral subgroup of $O(3)$
FITTING TO FINITE SCALING

\[ U[(\beta - \beta_{cr})L^{1/\nu}, (\lambda - \lambda_{cr})L^{-\omega},...] \simeq \]

\[ U^*(x) + O(L^{-\omega}) \simeq U^*(0) + a_1(\beta - \beta_{cr})L^{1/\nu} + c(\lambda)L^{-\omega} + \cdots \]

\[ U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2} \]

\[ \beta_{cross} \simeq \beta_{cr} + c_1 L^{-1/\nu-\omega} \]

\[ \beta_{crit} = 0.16098703(3) \]

Double Scaling: \[ x = (\beta - \beta_{cr})L^{1/\nu} \]
DETERMINE “SPEED OF LIGHT” VIA DESCENDANT RELATION & RESCALE “LOG(R)”

\[
\frac{(\mu_2 - \mu_1)}{\mu_1 - \mu_0} \text{ vs. } \frac{1}{s}
\]

\[
s/\Lambda = c + c'/s
\]

\[
c = 1.5105(7)
\]
CURRENT FIT:

Icosahedron(s), $T=8 \times s$, $\beta=0.16098700$

\[ \Delta_\sigma = 0.5 \left[ \frac{(\mu_1 + \mu_0)}{\mu_1 - \mu_0} - 1 \right] \]

$\Delta_\sigma \approx 0.5175$
1st Failure to Recover \( \text{fullo}(4,1) \) of \( L = 3 \)?

Apparent lack of convergence to a single \( \text{O}(3) \) irreducible representation for \( l = 3 \)
\[ \mathcal{L}(x) = -\frac{1}{2} (\nabla \phi)^2 + \lambda (\phi^2 - \mu/2\lambda)^2 \]

**Wilson-Fisher FP**

**Gaussian FP**

\[ \beta_g = \epsilon g - \frac{3}{16\pi^2} g^2 + O(g^3, \epsilon g^2, \mu^4, \mu^2 g; ) \]

\[ \beta_{\mu^2} = 2\mu^2 + ag + \frac{9}{16\pi^2} g\mu^2 + O(\mu^4) \]

\[ \lambda = 4g/4! \]
DISCRETIZE A LAGRANGIAN ON SIMPLICIAL MANIFOLD?

\[ L = \int d^3x \left[ \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda \sqrt{g} (\phi^2 - \mu^2 / 2\lambda)^2 \right] \]

project spherical triangle onto local tangent plane
Finite Element Method: What is it?

GOOGLE: 3,410,000 RESULTS
HRENNIKOFF, ALEXANDER (1941). COURANT, R. (1943)

RCB, M. Cheng and G.T. Fleming,
“Improved Lattice Radial Quantization” PoS Lattice 2013
“Quantum Finite Elements: 2D Ising CFT on a Spherical Manifold” Pos Lattice 2014

Saturday, April 25, 15
LINEAR FEM REFERENCES


• FEM: Discrete Exterior Calculus (de Rahm Complex, Whitney, etc., etc.), even ‘t Hooft, Luscher...
KINETIC TERM FOR LINEAR ELEMENT

On each triangle expand: \( \phi(x, y) = \sum_i K_i(x, y) \phi_i \) and integrate

\[
\int_{A_{012}} dxdy \partial_\mu \phi(x, y) \partial_\mu \phi(x, y) = \frac{1}{2A_{012}} [ (l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic} ]
\]
LINEAR FINITE ELEMENT METHOD FOR TRIANGULATE MANIFOLD

\[ g_{ij}(0) = \vec{e}_i \cdot \vec{e}_j \]

\[ K^\triangle_i(r) = \frac{A_i}{A_{123}} \]

\[ W_i(r) = \sum_\triangle K^\triangle_i(r) \]

\[ \phi(r) = \sum_i W_i(r) \phi_i \]

Piecewise linear subspace of Hilbert space

\[ d\vec{x} = \vec{e}_i dx^i \]

\[ ds^2 = d\vec{x} \cdot d\vec{x} = g_{ij} dx^i dx^j \]
**THEORY AND REGGE FORMALISM**

**Barycentric co-ord in a triangle**

\[ \xi_1 + \xi_2 + \xi_3 = 1 \quad , \quad 0 < \xi_i < 1 \]

\[ \vec{x} = \vec{x}_0 + \vec{e}_1 \xi_1 + \vec{e}_2 \xi_2 \]

\[ ds^2 = g_{ij} d\xi^i d\xi^j \]

\[ \vec{\nabla} = \vec{a}_i \partial_{\xi_i} \]

\[ \vec{e}_i \cdot \vec{a}_j = \delta_{ij} \]

**Linear interpolation**

\[ \phi(\xi_1, \xi_2) = \phi_0 + \xi_1 \phi_1 + \xi_2 \phi_2 \equiv \xi_1 \phi_1 + \xi_2 \phi_2 + \xi_3 \phi_0 \]

**Dual (reciprocal) vectors**

\[ \partial_{\mu} = e^j_{\mu} \partial_{\xi_j} \]
REGGE CALCULUS FORMULATION FOR SMOOTH MANIFOLD.

\[ FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2} \]

Delaunay Link Area: \( A_d = h_1 l_1 \)

\[ \sum_{\triangle kij} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}} \]

FEM GIVES “SPECTRAL FIDELITY”

- Taylor expansion on hypercubic lattice:

\[
a^{-1} \sum_{\pm \mu} (\phi(x) - \phi(x + a\mu))^2 \simeq (\nabla \phi)^2 + O(a^2)
\]

- Taylor series for FEM does not work!

\[
a^2 \sum_y K(x, y)(\phi(x) - \phi(y))^2 \simeq c_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + O(a^2)
\]

- FEM theorems: error & spectra < cut-off converges $O(a^2)$ if triangles are “shape regular” and “uniformly” refined.
FEM FIXES THE HUGE SPECTRAL DEFECTS

For $s = 8$ first $(l+1)(l+1) = 64$ ev

BEFORE ($K = 1$)

AFTER (FEM $K$'s)
SPECTRUM OF FE LAPLACIAN ON A SPHERE

$s = 8$

Fit

\[ l + 1.00012 l^2 - 13.428110^{-6} l^3 - 5.5724410^{-6} l^4 \]
2D ISING TEST CASE: PHI 4\textsuperscript{TH} ON THE RIEHMANN* SPHERE!

projection \[ \xi = \tan(\theta/2)e^{-i\phi} = \frac{x + iy}{1 + z} \]

*(1826–66), German mathematician; full name Georg Friedrich Bernhard Riemann. He founded Riemannian geometry, which is of fundamental importance to both mathematics and physics.
Exact Solution to CFT

Exact Two point function

\[
\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \rightarrow \frac{1}{|1 - \cos \theta_{12}|^{\Delta}}
\]

\[\Delta = \eta/2 = 1/8\]

4 pt function

\[
(x_1, x_2, x_3, x_4) = (0, \xi, 1, \infty)
\]

\[
g(0, \xi, 1, \infty) = \frac{1}{2|\xi|^{1/4}|1 - \xi|^{1/4}}[1 + \sqrt{1 - \xi} + |1 - \sqrt{1 - \xi}|]
\]

Critical Binder Commulant

\[
U_B^* = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \langle M^2 \rangle} = 0.567336
\]
UV DIVERGENCE BREAKS ROTATIONS

\[ \delta m^2 = \lambda \langle \phi(x) \phi(x) \rangle \rightarrow \frac{1}{K_{xx}} \]

one configuration

average of config.
NEED TO IMPROVE QUANTUM LAGRANGIAN
3 POSSIBLE SOLUTIONS?

(i) Pauli-Villars* 1949

\[
\frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} \equiv \frac{1}{p^2 + p^4/M_{PV}^2}
\]

(ii) Subtract x-dependent mass Counter term

\[
1/\xi \ll M_{PV} \ll \pi/\alpha
\]

(iii) Better simplex distribution (Exact density)

(*Richard Feynman, Ernst Stueckelberg)
$\delta m_0^2 = \frac{\sqrt{3}}{8\pi} \log(N/\sqrt{g_x}) + O(1/N)$

$\sqrt{g} = e^{\rho(x,y)} = \frac{(x^2 + y^2 + 1 - R^2)^{3/2}}{\sqrt{1 - R^2}}$

Dilation of FEM diffeomorphism: $g = e^{-2\rho(x)}$
BINDER CUMMULANT

Fits: $U_c = 0.5661(2) \text{ vs } 0.567336$

$\nu = 0.978(25) \text{ vs } 1$
\[ \int_{-1}^{1} dz \left( \frac{2}{1-z} \right)^{1/8} P_l(z) \]

\[ \Delta_\sigma = \eta/2 = 1/8 \simeq 0.128 \]

\[ \Rightarrow \quad \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{187963199}, \frac{541373840}{5450932771}, \ldots \]

Very fast cluster algorithm:
Brower, Tamayo ‘Embedded Dynamics for phi 4th Theory’ PRL 1989. Wolff
single cluster + plus Improved Estimators etc
Binder Cumulant with Analytic Counter Term

$d=2 \, \phi^4 \, \lambda=1.0$ analytic CT

exact: $U_c = 0.851003$
fit: $U_c = 0.8529(1)$
ANALYTIC VS NUMERICAL CT

Conical Singularity?
FERMIIONS AND GAUGE FIELDS
“FEM” OR DISCRETE EXTERIOR CALCULUS

Fermions on smooth manifold:
\[ e^\mu(x) = e^\mu_a \gamma^a \]
\[ \bar{\psi} e^\mu(x) (\partial_\mu - \omega_\mu) \psi \]
\[ \omega_\mu(x) = \omega_\mu^{ab} \gamma^a \gamma^b \]
\[ g_{\mu\nu}(x) = e^a_\mu(x) e^a_\nu(x) \]

(1) **Lattice Fermions** are on simplicial curved manifolds must be done with great care: Spin connection does not always exist! (Doing c = 1/2 fermion on sphere)

(2) **Compact gauge links** can be using Nedellic/Whitney “edge” elements the spirit of Christ, Friedberg, Lee & G. ‘tHooft

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WHAT ABOUT RADIAL QUANTIZATION OF YANG MILLS? (EVEN

\[ \mathbb{R} \times T^3 \] vs \[ \mathbb{R} \times S^3 \]

1. Short distances: Asymptotically free OPE
2. In confining phase: Hamiltonian evolution!
3. In Conformal Window: Dilatation evolution!
4. Relevant mass deformation

(Spectral flow: Anomalous Dimension ➔ Energies!)
Radial Quantization: *Early History*

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

**Abstract:** A field theory is quantized covariantly on Lorentz-invariant surfaces. *Dilatations replace time translations* as dynamical equations of motion. .... The *Virasoro algebra of the dual resonance model* is derived in a wide class of 2-dimensional Euclidean field theories.


**Abstract:** The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. *For d > 2 these correspond however, to curved spaces.* The result is verified for the spherical model.
Proof of one loop QFEM Counter Term:

Try: \[ \delta m^2 = q(x) \log\left(\frac{1}{a^2 m^2}\right) + c(x) + O(a^2) \]

- **Step #1:** FEM: Cea’s lemma (no)

\[ \| \phi(x) - \phi_h^{fem}(x) \| < \| \phi(x) - v_h(x) \| \]

where \[ \| v(x) \| = \int_\Omega dx [\nabla v(x) \cdot \nabla v(x) + m^2 v^2(x)] \]

\[ \implies q(x) = Q \]

- **Step #2:** Regge Gravity: diffeomorphisms

\[ dzd\bar{z} = |f'(w)|^2 dwd\bar{w} \implies Q \log(|w|^2) = Q \log(|z|^2 / \sqrt{g(z)}) \]
WEYL VS CONFORMAL DIFFEOMORPHISMS

• Weyl (change the manifold) \[ g_{\mu\nu}(x) \rightarrow \Omega(x)g_{\mu\nu}(x) \]

• Diffeomorphism (change the co-ordinates) \[ x \rightarrow \xi = f(x) \]

• Conformal Diffeomorphism (largest subgroup)

• Primary field: \[ \phi(\xi) = b^{-\Delta}(x)\phi(x) \]

\[ ds^2 = g_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu \]

\[ ds^2 = b^2(x)g_{\mu\nu}(x)dx^\mu dx^\nu \]

Theorem: \[ |\xi_1 - \xi_2|^2 = b(x_1)|x_1 - x_2|^2b(x_2) \]
SIMPLICIAL LATTICE INDUCES A DIFFEOMORPHISM

• We have found the counter term of the FEM Lagrangian
• Our sequence of simplicies induce a (conformal?) diffeomorphism
• BUT now need to find correct for primary operator:

\[ \phi(x) \rightarrow \Phi(x) = b(x)^{-\Delta} \phi(x) \]

• This is now being implement (Stay tuned)
• CT for 3D are being computed and will be tested
SCALING VS FULL CONFORMAL SYMMETRY

- General Field Theory with Scale invariance and Poincare Invariance

- $O(d) \implies O(d,1)$ (Isometries of AdS space)

\[
x_\mu \to \lambda x_\mu , \quad x_\mu \to \frac{x_\mu}{x^2}
\]

\[
K_\mu : (\text{inv} \to \text{trans} \to \text{inv})
\]

\[
[K_\mu, \mathcal{O}(x)] = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)
\]

\[
[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)
\]

\[
[D, P_\mu] = -iP_\mu , \quad [D, K_\mu] = +iK_\mu , \quad [K_\mu, P_\mu] = 2iD
\]
Einstein Regge Curvature

$$\delta_v = 2\pi - \sum_{i \in V} \theta_i$$

$$\sum_v \delta_v = 2\pi \chi = 2\pi (F - E + V)$$

Could Optimize adaptive Delaney triangles on unit sphere

$$\int d^2 x \sqrt{g} [\lambda - kR^2 + aR^2] \implies \sum_v A_v [\lambda - 2kR_v + aR_v^2]$$

$$R_v = 2\delta_v / A_v$$  \hspace{1cm} \text{flat triangles: } \delta_v = 4\pi / A_v$$