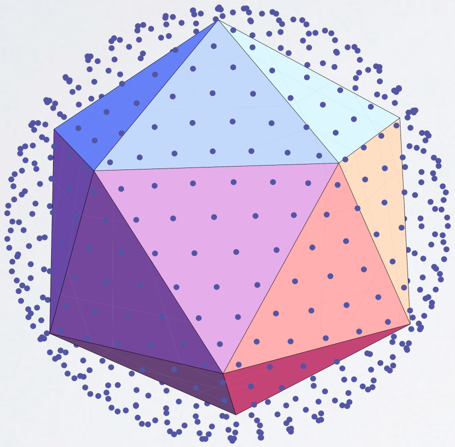
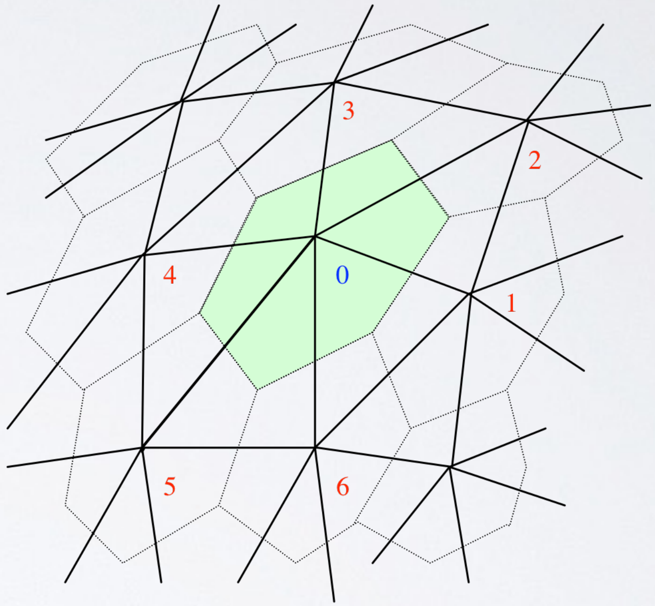
TEST OF QFEM*:
2D ISING CFT ON RIEMANN SPHERE





Richard Brower

BSM at Livermore Lab, April 25, 2015

*Quantum Finite Element Method

HISTORY: WHERE AM I ?

(5+ YEARS AFTER BIRTH OF LATTICE QCD)

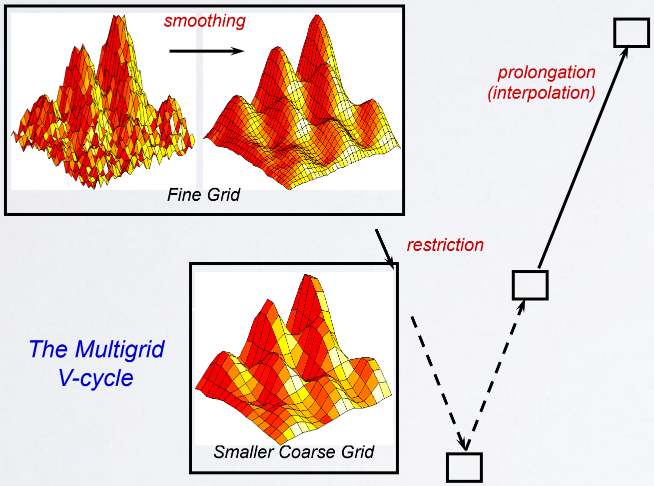
Courtesy Special Collections, UC Santa Cruz

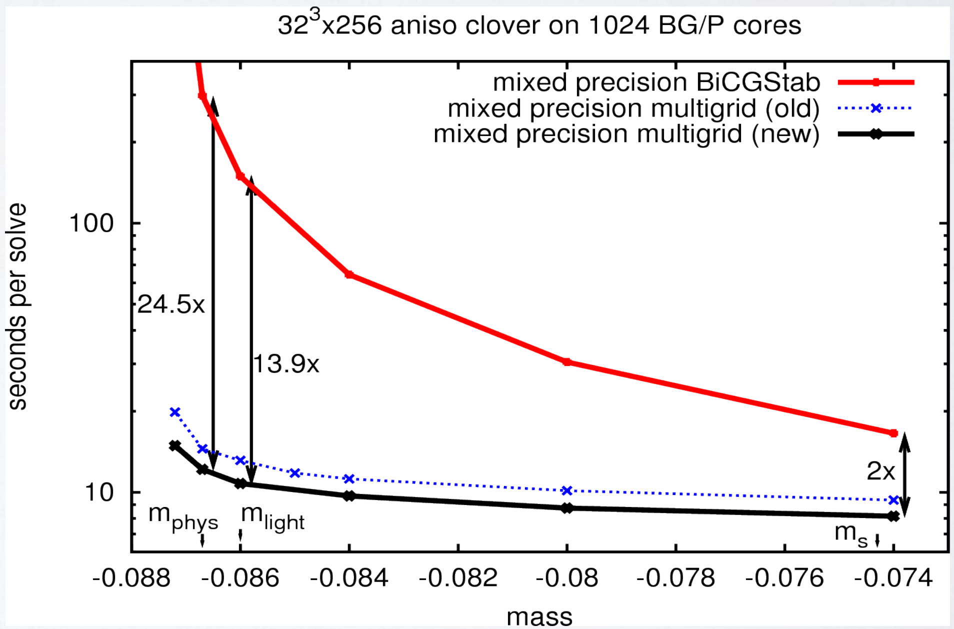
20+ Years to adapt RG for Multigrid Lattice Dirac Operators

Journal of Computational Physics Volume 80, Issue 2, February 1989, Pages 472–479

Combining renormalization group and multigrid methods (Brower, Giles, Moriarty, Tamayo)

"Adaptive multigrid algorithm for the lattice Wilson-Dirac operator" R. Babich, J. Brannick, R. C. Brower, M. A. Clark, T. Manteuffel, S. McCormick, J. C. Osborn, and C. Rebbi, PRL. (2010).





ADAPTIVE SMOOTH AGGREGATION ALGEBRAIC MULTIGRID

OUTLINE

Motivation: Conformal Fixed Point

Curved Manifolds via FEM + Regge (IR)

Counter Terms for Multiple UV cut-off.

Test Case: c= 1/2 2D CFT (e.g. Ising)

Future Tasks. 3D, Fermions, Gauge Theory etc.

Motivation for FEM Lattice Field Theory

- Conformal Field Theories, interesting for
  - BSM composite Higgs
  - AdS/CFT weak-strong duality
  - Model building & Critical Phenomena in general

Potential Huge Advantage for CFT!

- Linear Hypercubic vs Exponential Radial Lattice

$$a < r < aL \to 1 < \log(r) < L$$

Both UV asy freedom and IR conformal on a lattice?

Radial Quantization $\mathbb{R}^d \Longrightarrow \mathbb{R} \times \mathbb{S}^{d-1}$

Evolution: $H = P_0 \text{ in } t \Longrightarrow D \text{ in } \tau = \log(r)$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$\text{"time"} \quad \tau = log(r) \; , \quad \text{"mass"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

EXACT CFT: POWER LAW CORRELATOR

Conformal correlator: $\langle\phi(x_1)\phi(x_2)\rangle = C\dfrac{1}{|x_1 - x_2|^{2\Delta}}$

$$
\begin{aligned}
r_1^\Delta r_2^\Delta\langle\phi(\tau_1,\Omega_1)\phi(\tau_2,\Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^\Delta} \\
&\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\
&= Ce^{-\tau\Delta}
\end{aligned}
$$

With $|x_1 - x_2|^2 = r_1 r_2[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

BACK TO THE BOOTSTRAP! (CFTS : NO LOCAL LAGRANGIAN)

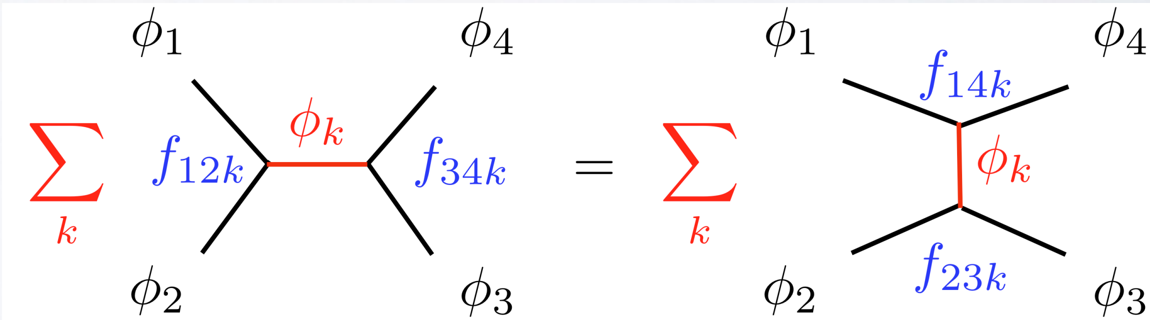
(i.e. Data: spectra + couplings to conformal blocks)

Exact 2 and 3 correlators

$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

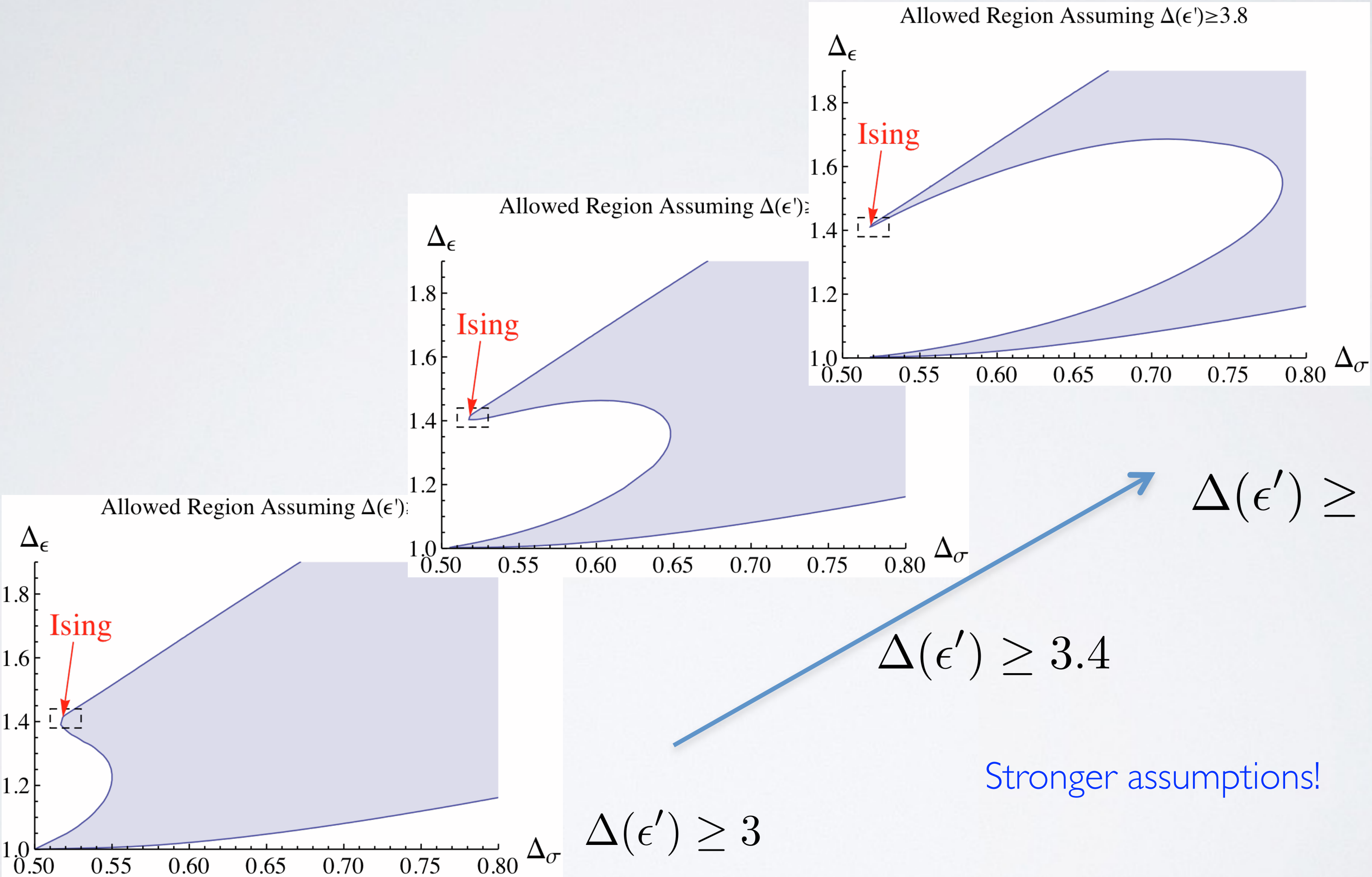
$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks



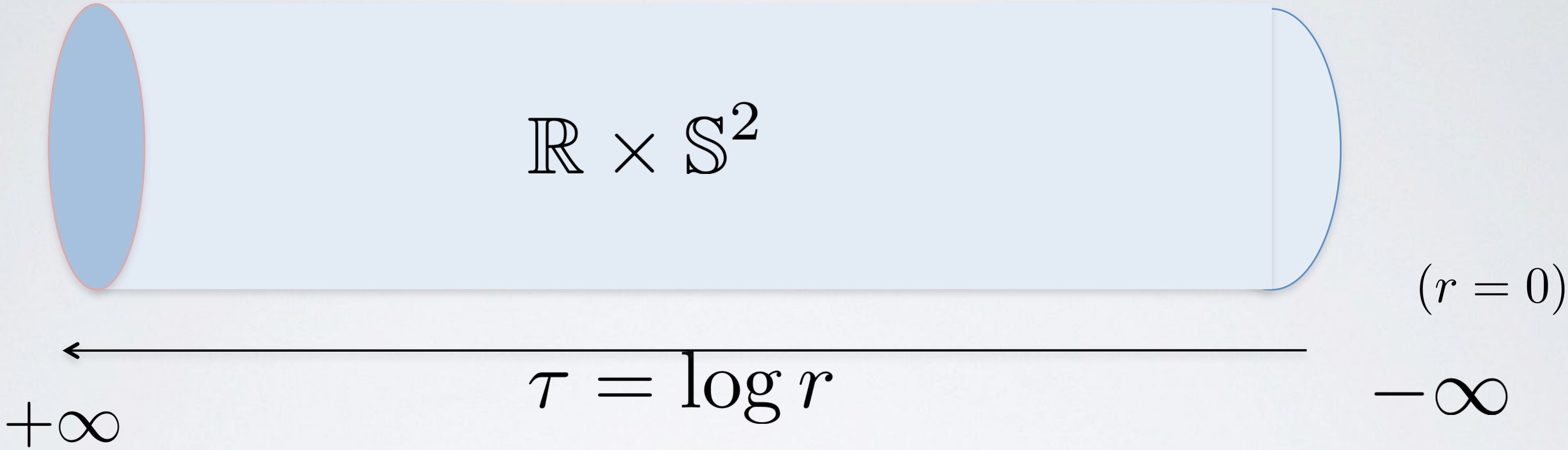
CFT Bootstrap: OPE & factorization completely fixed the theory

*INEQUALITIES FROM BOOTSTRAP**



- "Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

FIRST TOY PROBLEM: 3-D ISING AT WILSON-FISHER



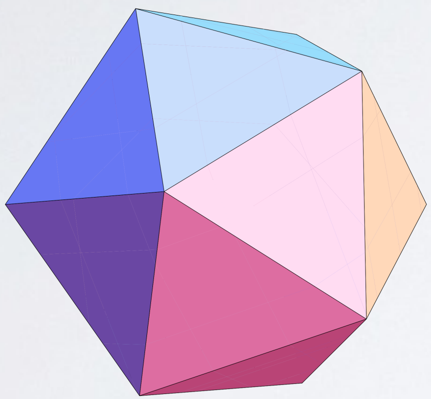
$$Z_{Ising} = \sum_{\sigma(x,t)=\pm 1} e^{\beta \sum_{t,\langle x,y\rangle} \sigma(t,x)\sigma(t,y) + \beta \sum_{t,x} \sigma(t+1,x)\sigma(t,x)}$$

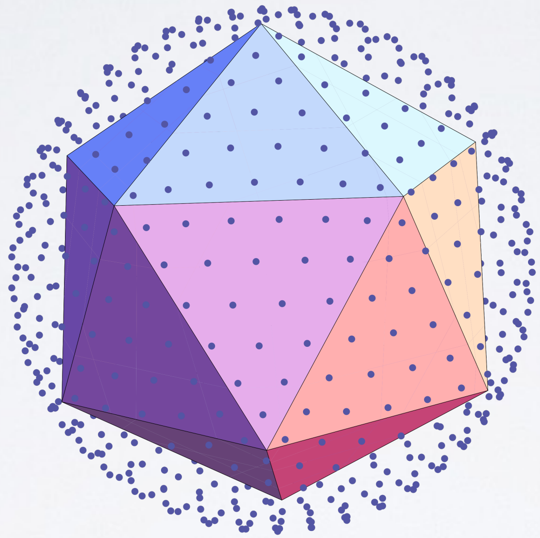
RCB, G. Fleming, H. Neuberger Phys.Lett, B721 (2013)

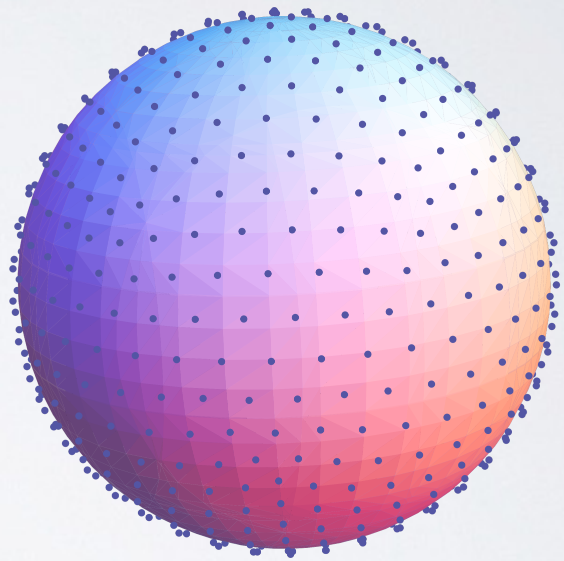
ORDER S REFINED TRIANGULATED ICOSAHEDRON

$s = 8$

$s = 1$



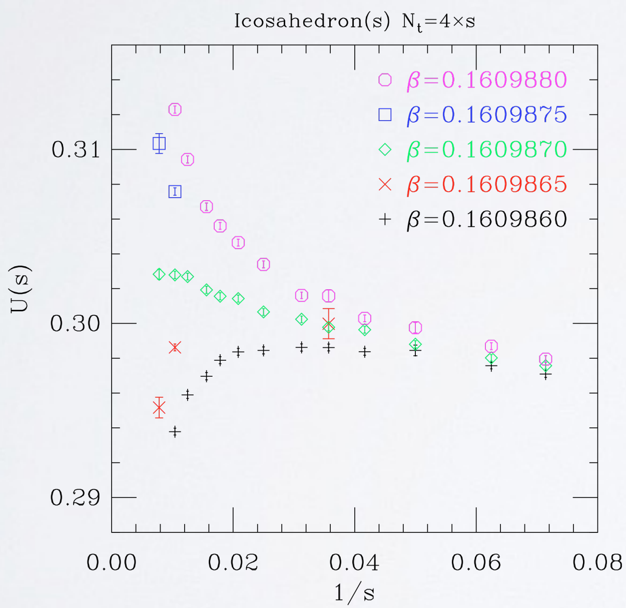




l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Iscosahedral subgroup of O(3)

FITTING TO FINITE SCALING

$$U[(\beta-\beta_{cr})L^{1/\nu},(\lambda-\lambda_{cr})L^{-\omega},\ldots]\simeq$$
$$U^*(x)+O(L^{-\omega})\simeq U^*(0)+a_1(\beta-\beta_{cr})L^{1/\nu}+c(\lambda)L^{-\omega}+\cdots$$



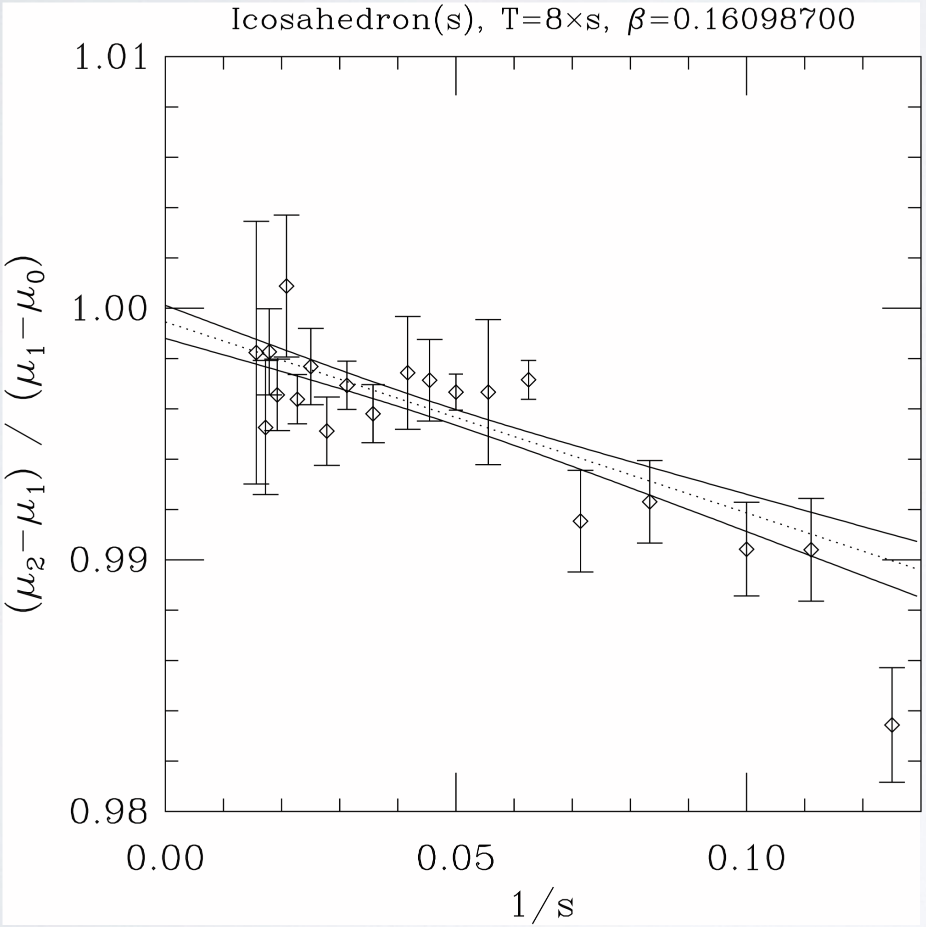
$$U(\beta,s=L)=1-\frac{\langle M^4\rangle}{3\langle M^2\rangle^2}$$

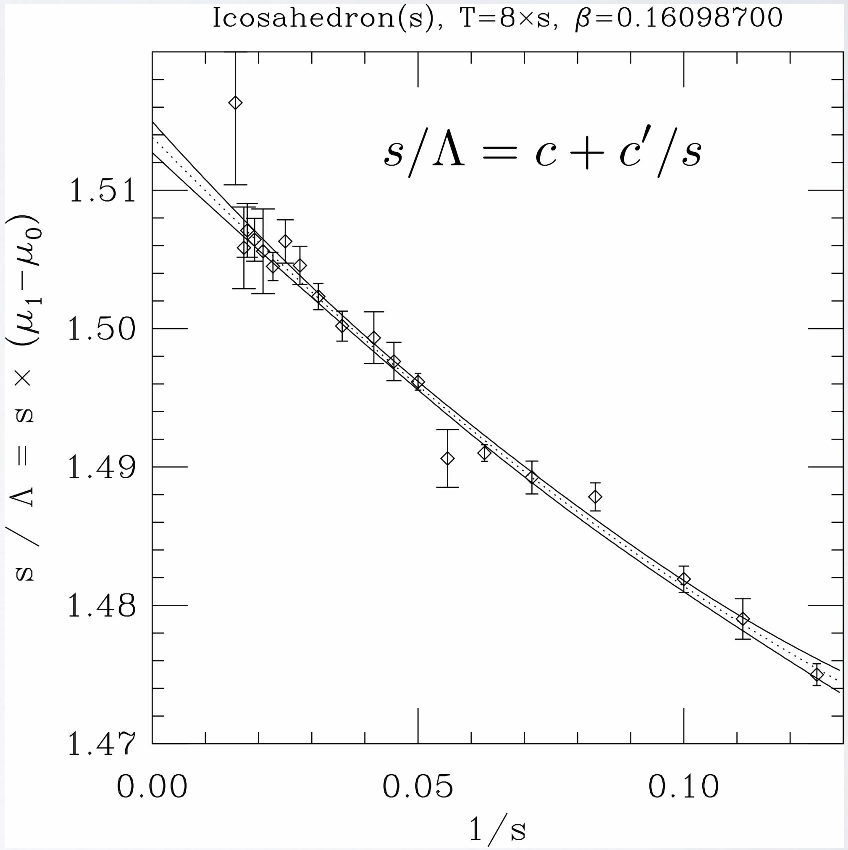
$$\beta_{cross}\simeq\beta_{cr}+c_1L^{-1/\nu-\omega}$$

$$\beta_{crit}=0.16098703(3)$$

Double Scaling: $x=(\beta-\beta_{cr})L^{1/\nu}$

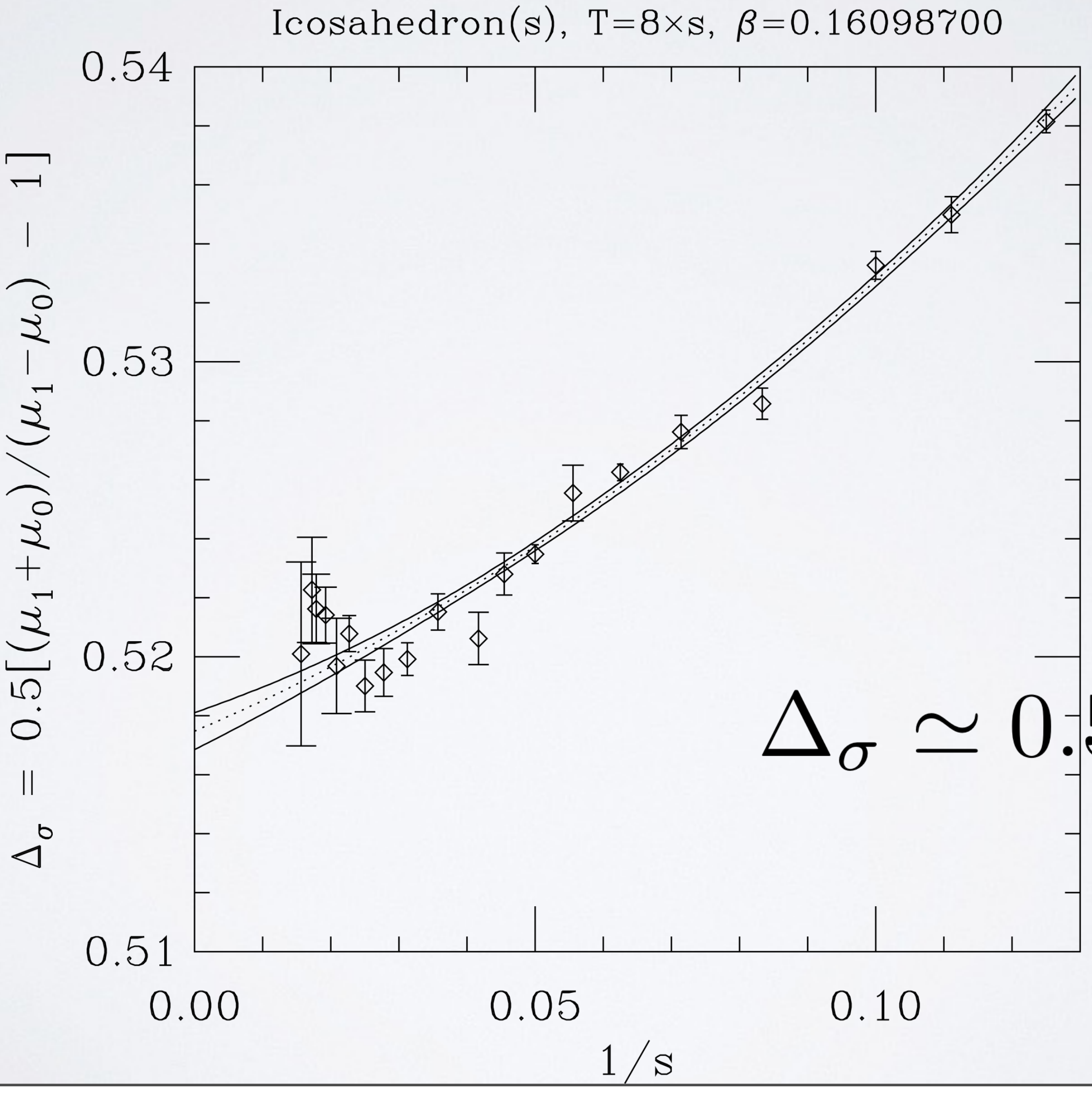
DETERMINE "SPEED OF LIGHT" VIA DESCENDANT RELATION & RESCALE "LOG(R)"





$$c = 1.5105(7)$$

CURRENT FIT:

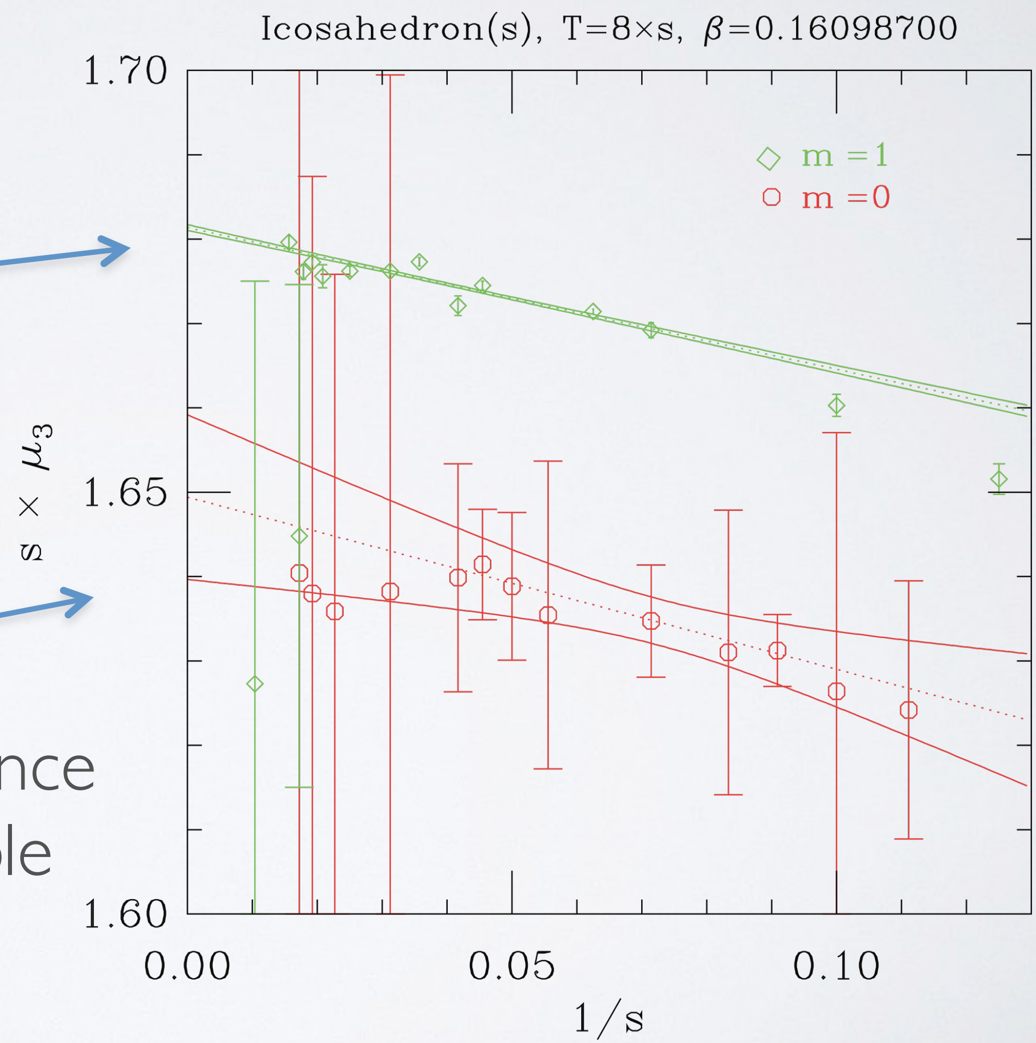


$\Delta_\sigma \simeq 0.5175$

1ST FAILURE TO RECOVER FULLO(4,1) OF L = 3?

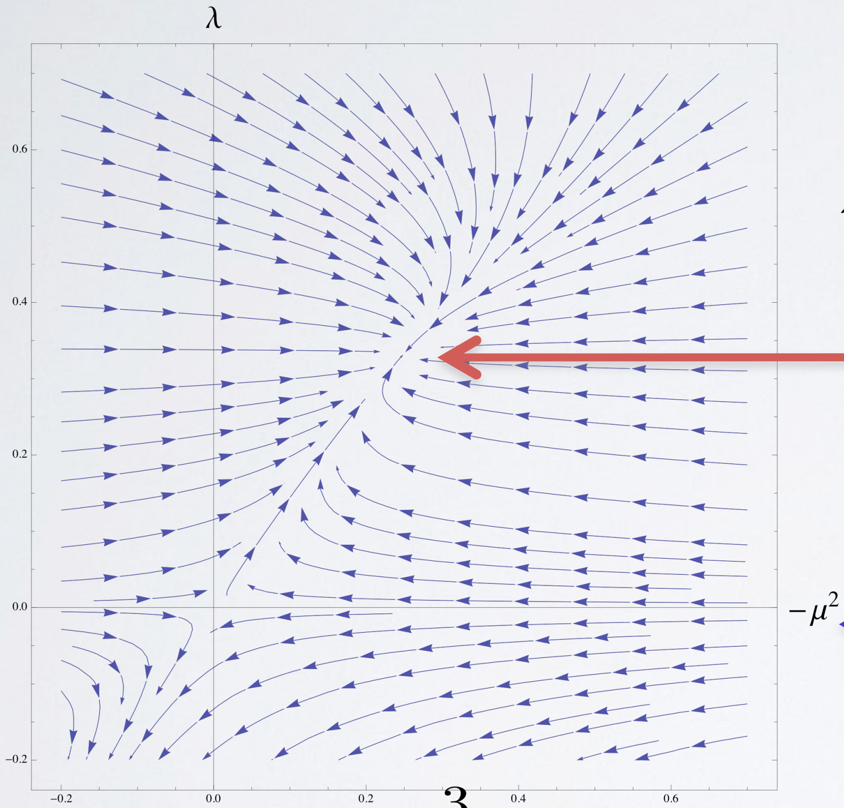
G rep

T2 rep



Apparent lack of convergence to a single O(3) irreducible representation for l = 3

USE FEM FOR PHI 4TH



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu/2\lambda)^2$$

Wilson-Fisher FP

Gaussian FP

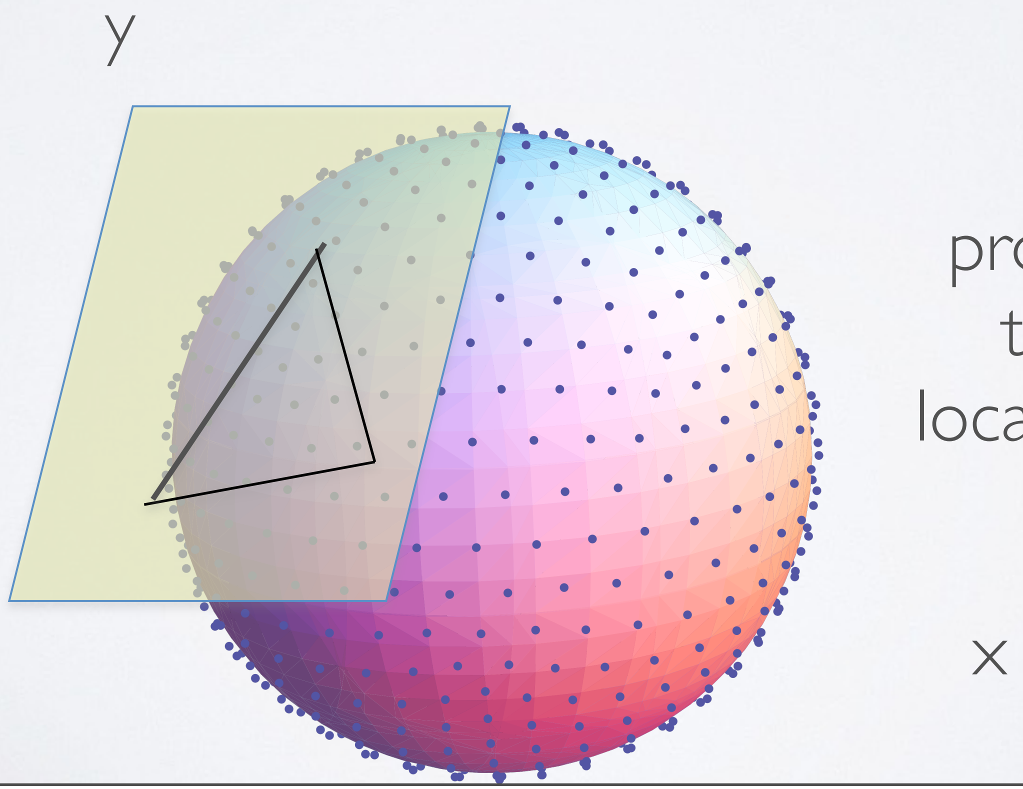
$$\beta_g = \epsilon g - \frac{3}{16\pi^2} g^2 + O(g^3, \epsilon g^2, \mu^4, \mu^2 g;)$$

$$\lambda = 4g/4!$$

$$\beta_{\mu^2} = 2\mu^2 + ag + \frac{9}{16\pi^2} g\mu^2 + O(\mu^4)$$

DISCRETIZE A LAGRANGIAN ON SIMPLICIAL MANIFOLD?

$$L = \int d^3x[\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda\sqrt{g}(\phi^2 - \mu^2/2\lambda)^2]$$

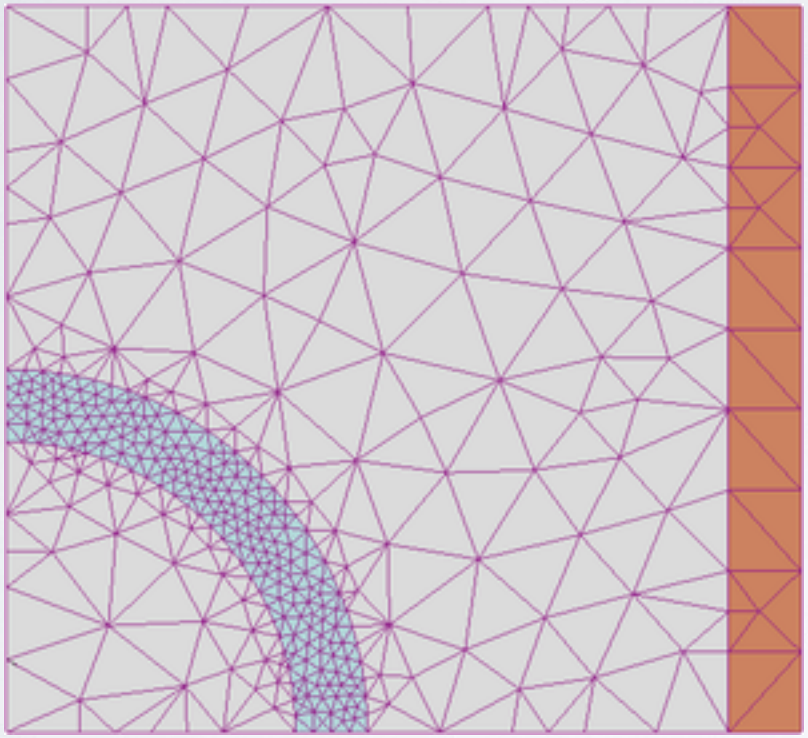


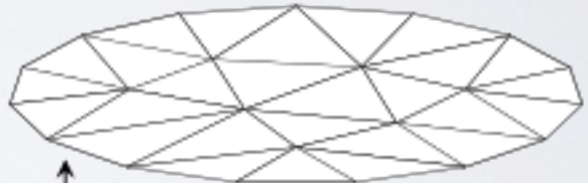
project spherical triangle onto local tangent plane

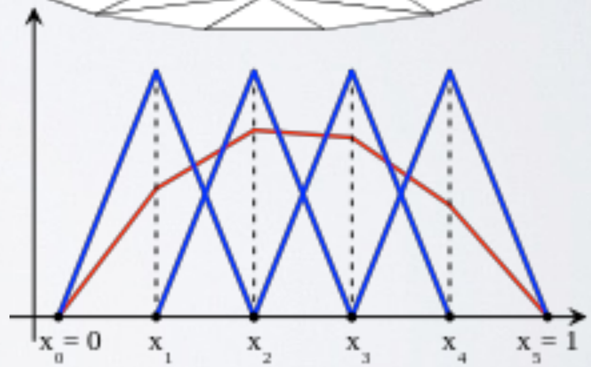
Finite Element Method: What is it?

GOOGLE: 3,410,000 RESULTS

HRENNIKOFF, ALEXANDER (1941). COURANT, R. (1943)







RCB, M. Cheng and G.T. Fleming,

“Improved Lattice Radial Quantization” PoS Lattice 2013

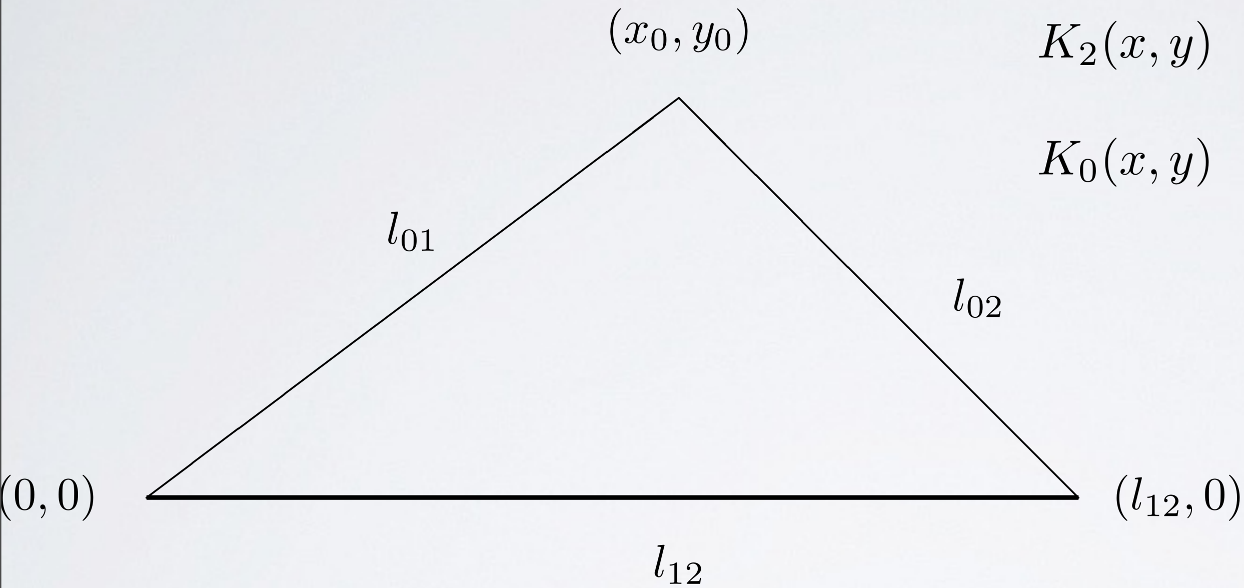
“Quantum Finite Elements: 2D Ising CFT on a Spherical Manifold” Pos Lattice 2014

LINEAR FEM REFERENCES

- Regge Calculus T. Regge, Nuovo Cimento 19 (1961) 558.
- Random Lattices: N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. B 202, 89 (1982).
- FEM: Discrete Exterior Calculus (de Rahm Complex, Whitney, etc, etc.), even ‘t Hooft, Luscher...

KINETIC TERM FOR LINEAR ELEMENT

$$
\begin{aligned}
K_1(x,y) &= [l_{12} - x - \frac{(l_{12}-x_0)y}{y_0}]/l_{12} \\
K_2(x,y) &= [x - \frac{x_0 y}{y_0}]/l_{12} \\
K_0(x,y) &= \frac{y}{y_0}
\end{aligned}
$$

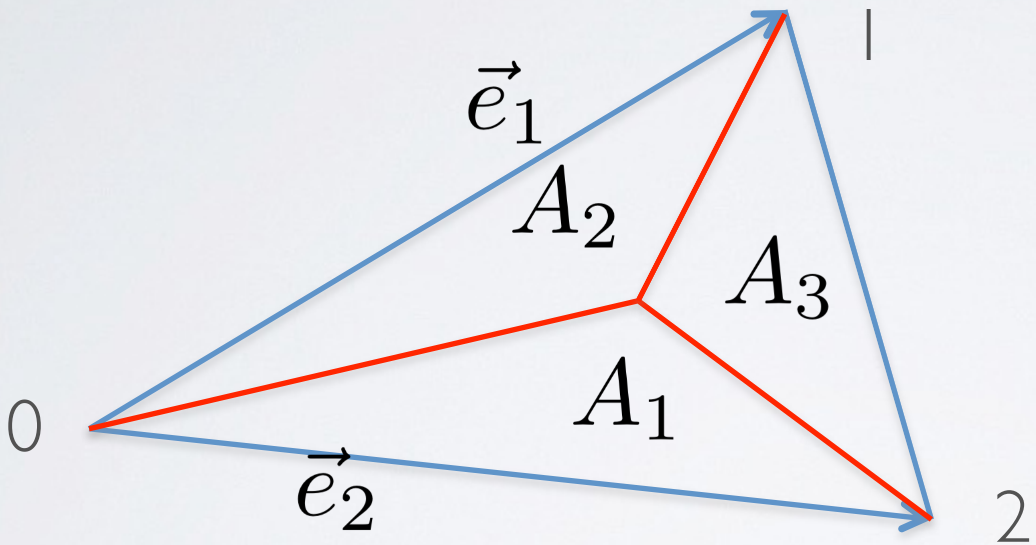


On each triangle expand: $\phi(x,y) = \sum_i K_i(x,y)\phi_i$ an integrate

$$
\int_{A_{012}} dx dy \partial_\mu \phi(x,y) \partial_\mu \phi(x,y) = \frac{1}{2A_{012}}[(l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}]
$$

LINEAR FINITE ELEMENT METHOD FOR TRIANGULATE MANIFOLD

$$g_{ij}(0) = \vec{e}_i \cdot \vec{e}_j$$



$$K_i^{\triangle}(r) = \frac{A_i}{A_{123}}$$

$$W_i(r) = \sum_{\triangle} K_i^{\triangle}(r)$$

$$\phi(r) = \sum_i W_i(r)\phi_i$$

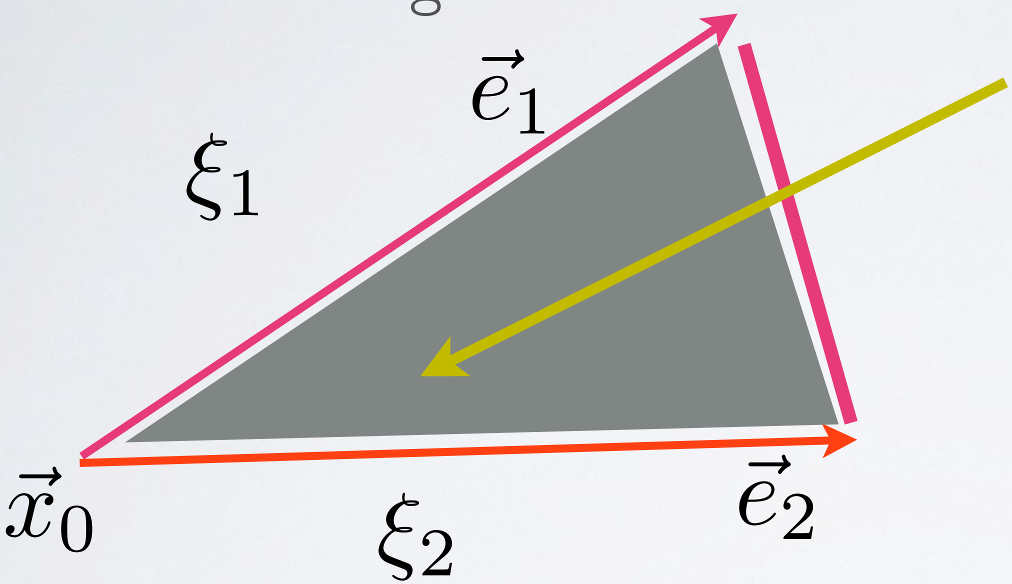
Piecewise linear subspace of Hilbert space

$$d\vec{x} = \vec{e}_i dx^i \qquad ds^2 = d\vec{x} \cdot d\vec{x} = g_{ij} dx^i dx^j$$

THEORY AND REGGE FORMALISM

Barycentric co-ord in a triangle

$$\xi_1 + \xi_2 + \xi_3 = 1 \quad , \quad 0 < \xi_i < 1$$



$$\vec{x} = \vec{x}_0 + \vec{e}_1 \xi_1 + \vec{e}_2 \xi_2$$

$$ds^2 = g_{ij} d\xi^i d\xi^J$$

$$\vec{\nabla} = \vec{a}_i \partial_{\xi^i}$$

$$\vec{e}_i \cdot \vec{a}_j = \delta_{ij}$$

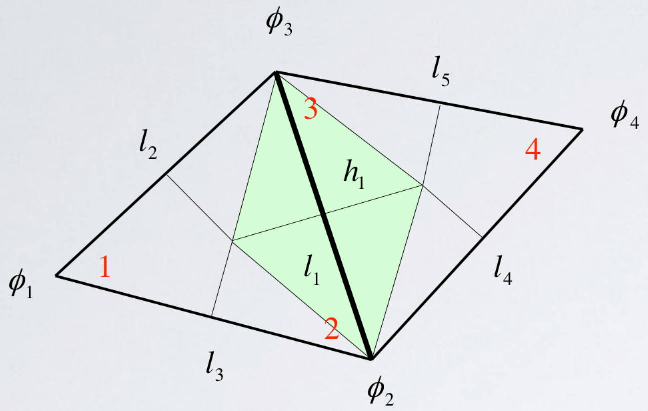
dual (reciprocal) vectors

$$\partial_\mu = e^j_\mu \partial_{\xi_j}$$

Linear interpolation

$$\phi(\xi_1, \xi_2) = \phi_0 + \xi_1 \phi_1 + \xi_2 \phi_2 \equiv \xi_1 \phi_1 + \xi_2 \phi_2 + \xi_3 \phi_0$$

REGGE CALCULUS FORMULATION FOR SMOOTH MANIFOLD.



$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

Delaunay Link Area: $A_d = h_1 l_1$

$$\sum_{\triangle_{kij}} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

H. Hamber, S. Liu, **Feynman rules for simplicial gravity,** NP B475 (1996)

FEM GIVES "SPECTRAL FIDELITY"

- Taylor expansion on hypercubic lattice:

$$a^{-1} \sum_{\pm\mu} (\phi(x) - \phi(x + a\mu))^2 \simeq (\nabla \phi)^2 + O(a^2)$$

- Taylor series for FEM does not work!

$$a^2 \sum_y K(x, y)(\phi(x) - \phi(y))^2 \simeq c_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + O(a^2)$$

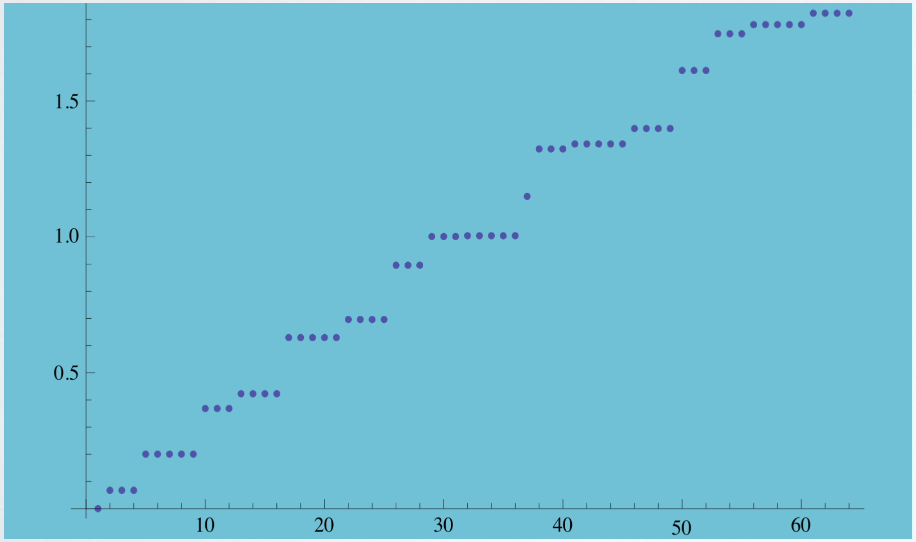
- FEM theorems: error & spectra < cut-off converges O(a^2) if triangles are "shape regular" and "uniformly" refined.

FEM FIXES THE HUGE SPECTRAL DEFECTS

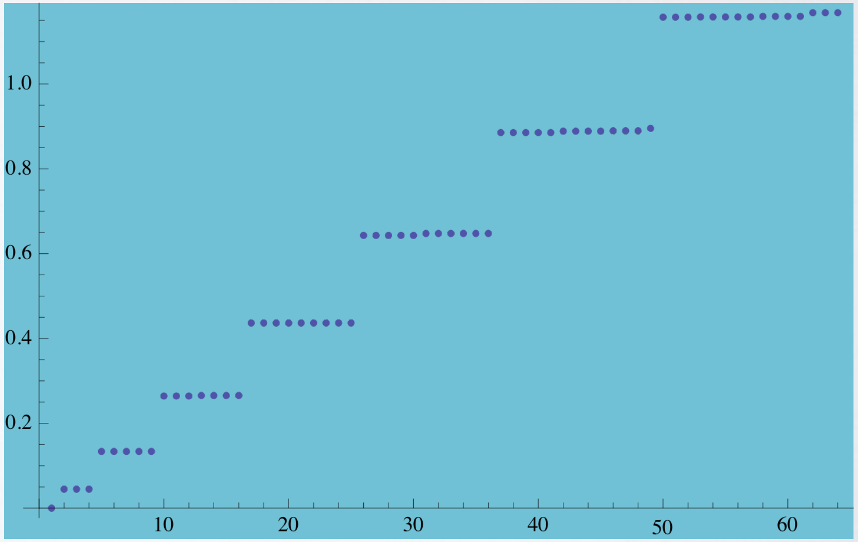
For s = 8 first (l+1)*(l +1) = 64 ev

BEFORE (K = 1)

AFTER (FEM K's)

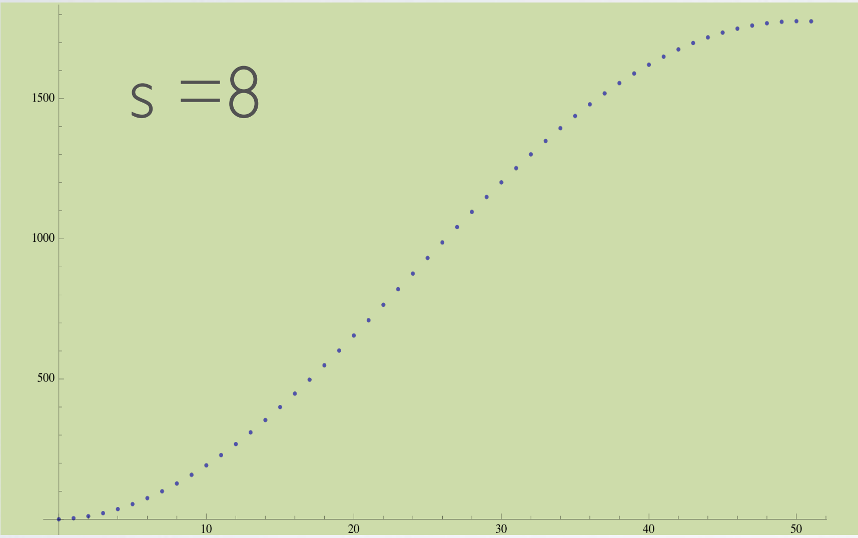


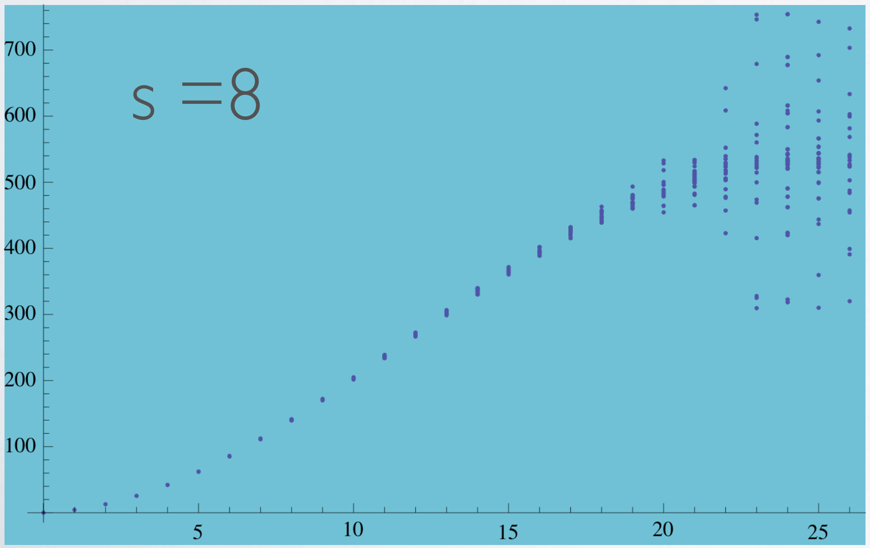
l, m

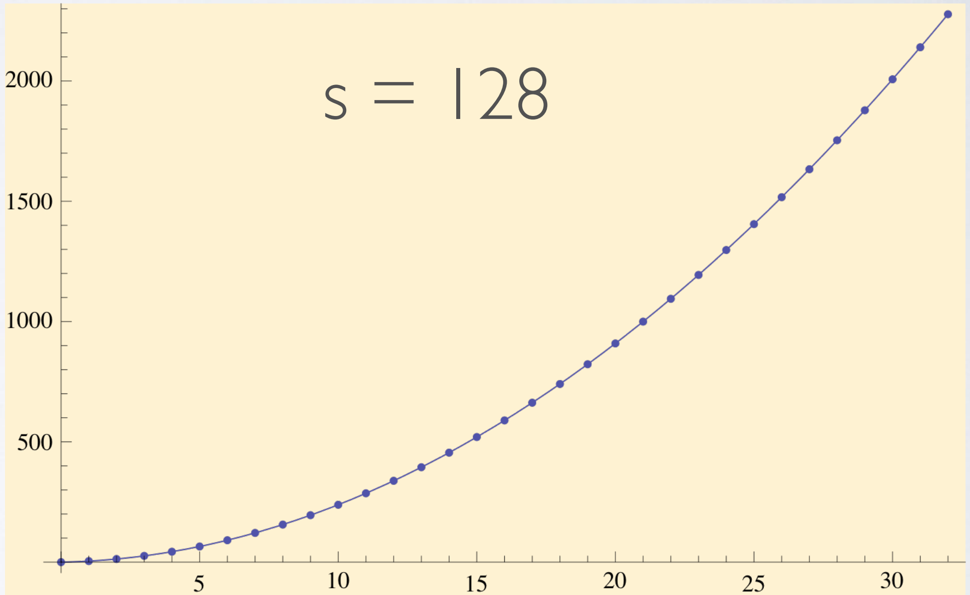


l, m

SPECTRUM OF FE LAPLACIAN ON A SPHERE





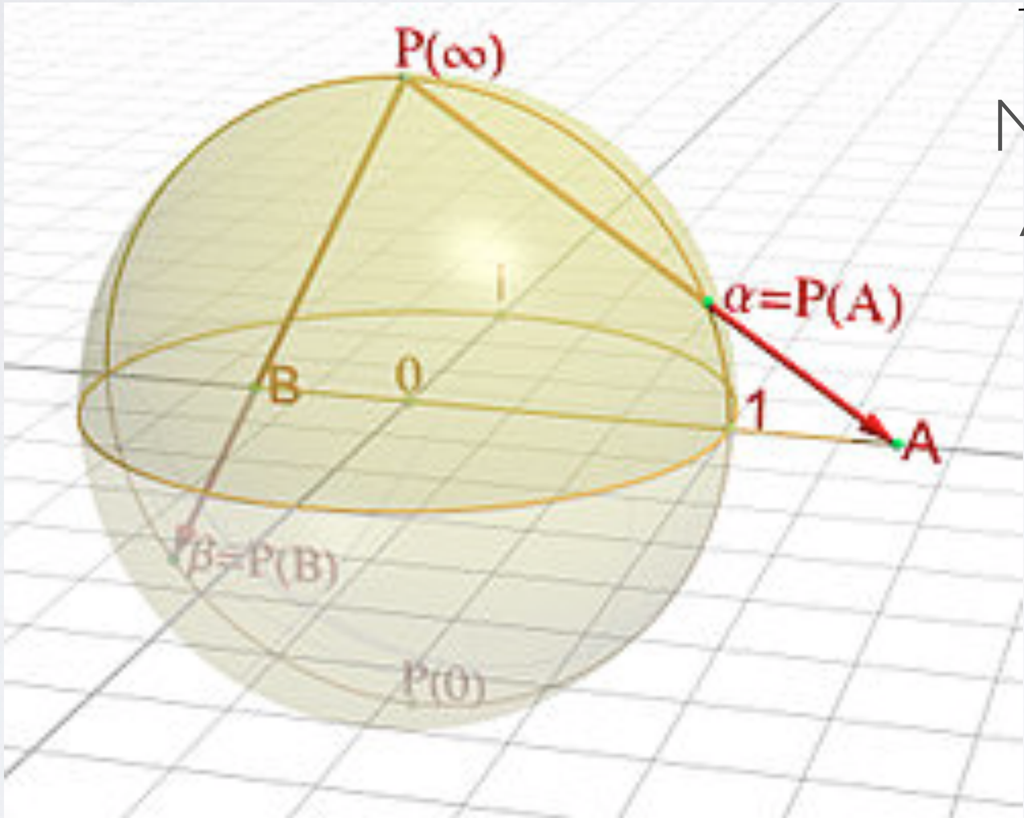


Fit

$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

2D ISING TEST CASE:
PHI 4^{TH} ON THE RIEMANN* SPHERE!

projection $$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$



NO FINITE VOLUME APPROXIMATION!

*(1826–66), German mathematician; full name Georg Friedrich Bernhard Riemann. He founded Riemannian geometry, which is of fundamental importance to both mathematics and physics.

EXACT SOLUTION TO CFT

Exact Two point function

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \rightarrow \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8$$

$$x^2 + y^2 + z^2 = 1$$

4 pt function

$$(x_1, x_2, x_3, x_4) = (0, \xi, 1, \infty)$$

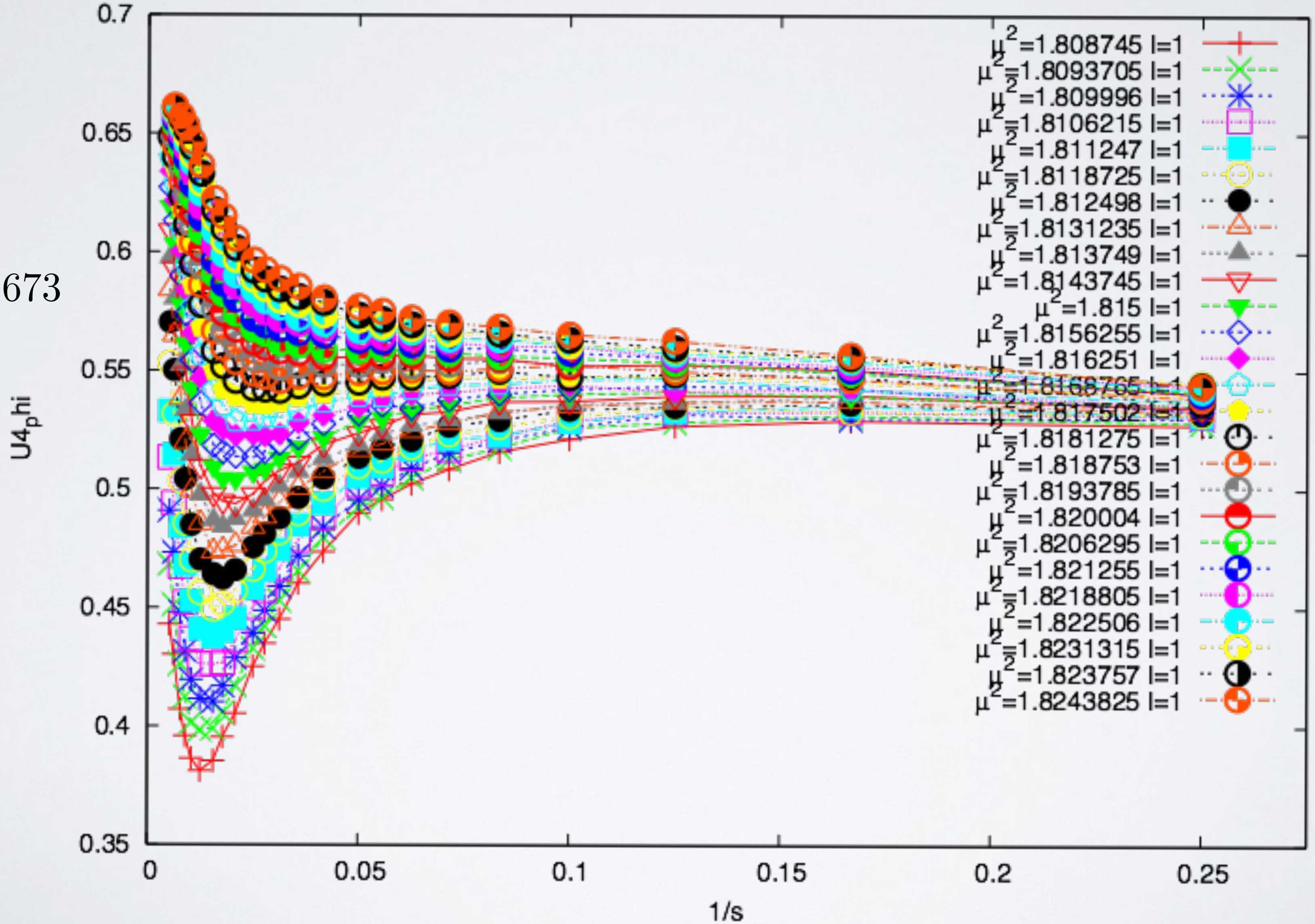
$$g(0, \xi, 1, \infty) = \frac{1}{2|\xi|^{1/4}|1-\xi|^{1/4}}[1 + \sqrt{1-\xi}| + |1 - \sqrt{1-\xi}||$$

Critical Binder Commulant

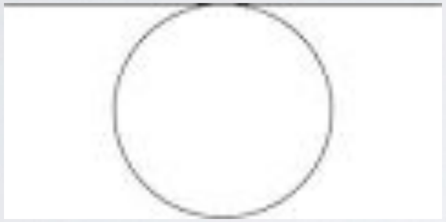
$$U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$$

BINDER CUMMULANT NEVER

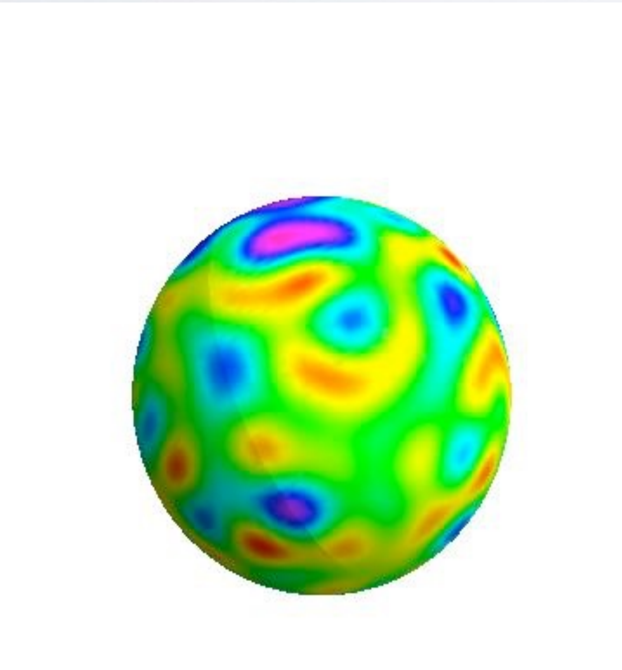
0.5673



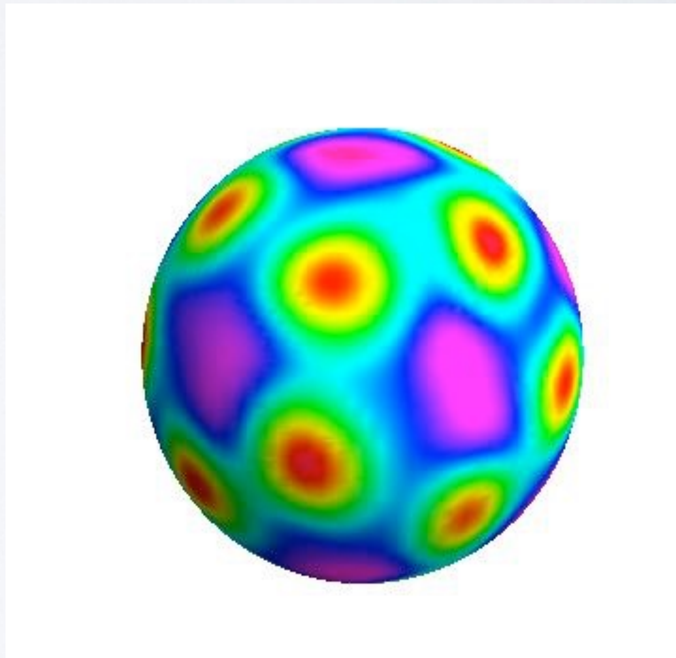
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda\langle\phi(x)\phi(x)\rangle \to \frac{1}{K_{xx}}$$



one configuration



average of config.

NEED TO IMPROVE QUANTUM LAGRANGIAN
3 POSSIBLE SOLUTIONS?

(i) Pauli-Villars* 1949

$$\frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} \equiv \frac{1}{p^2 + p^4/M_{PV}^2}$$

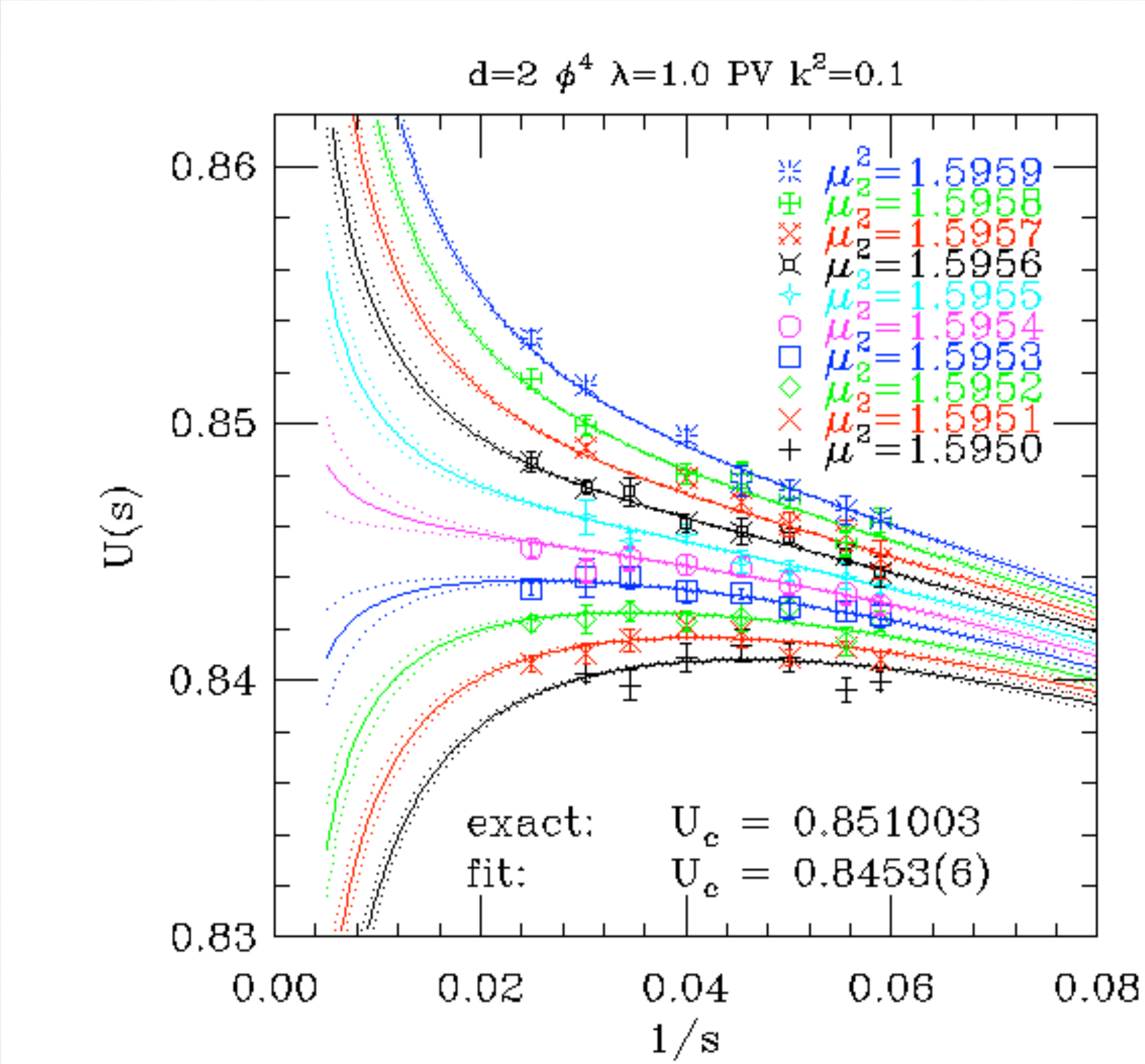
(ii) Subtract x-dependent mass Counter term

$$1/\xi \ll M_{PV} \ll \pi/a$$

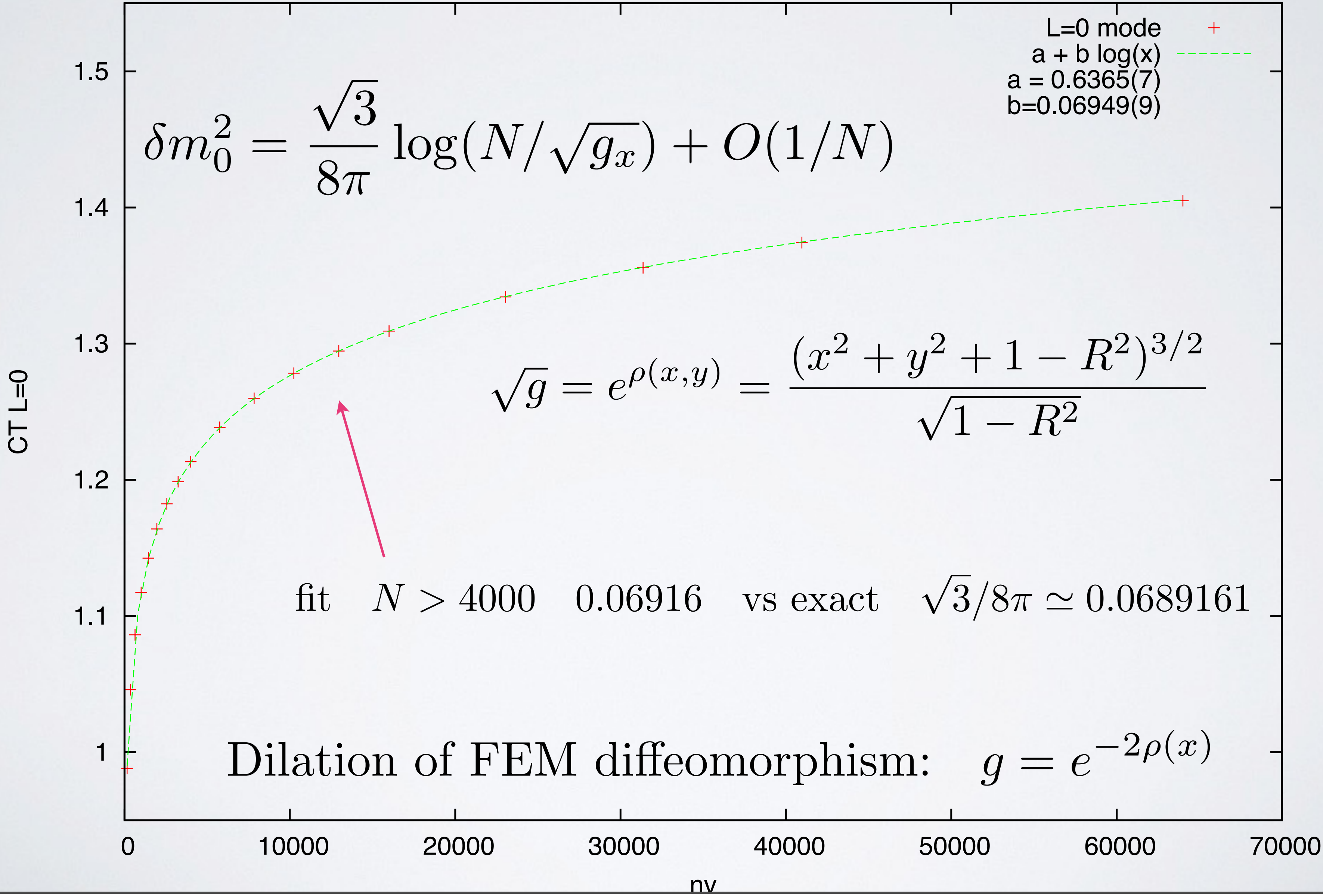
(iii) Better simplex distribution (Exact density)

*(*Richard Feynman, Ernst Stueckelberg)*

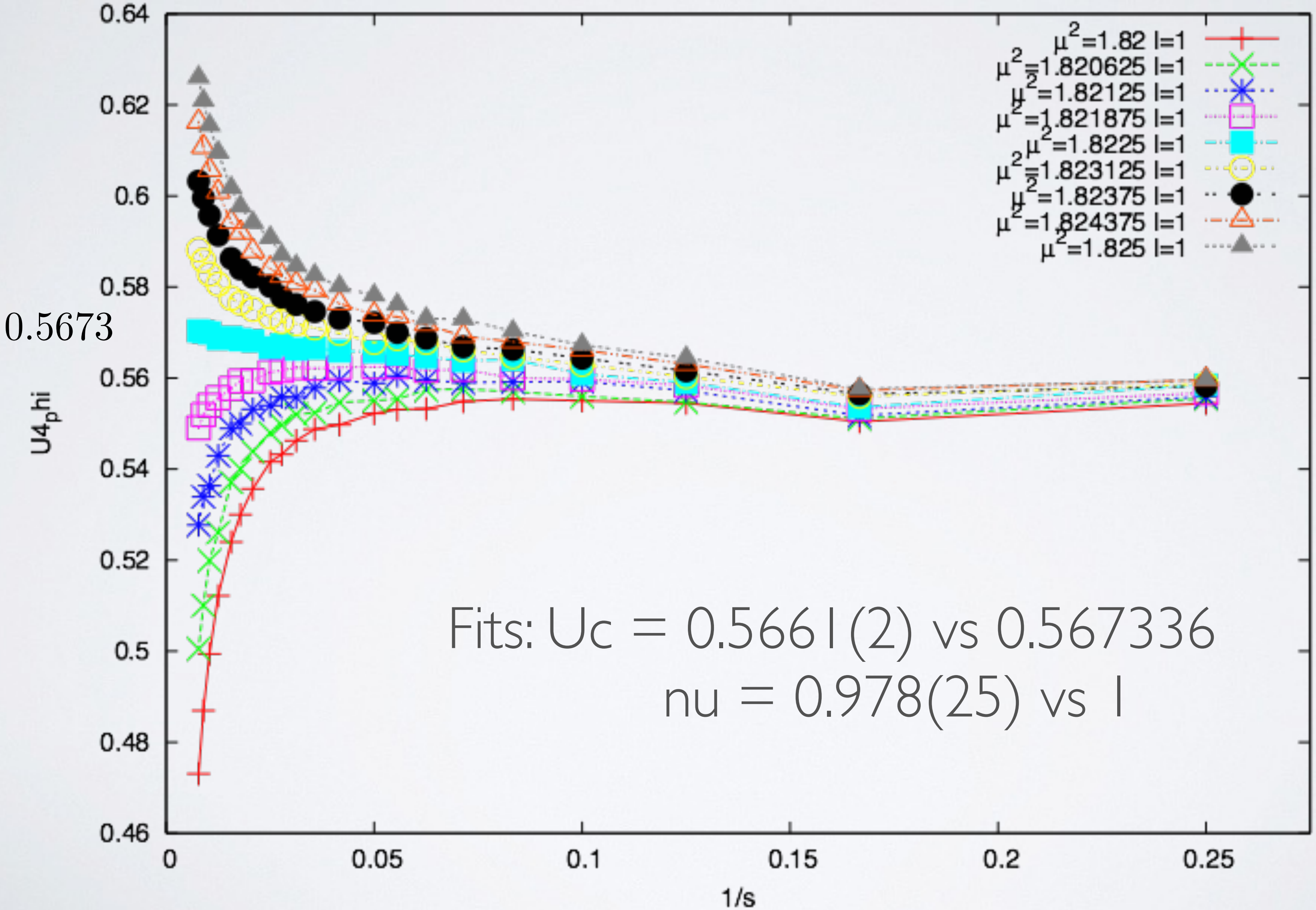
PAULI VILLARS



ANALYTIC FORM OF COUNTERTERM!

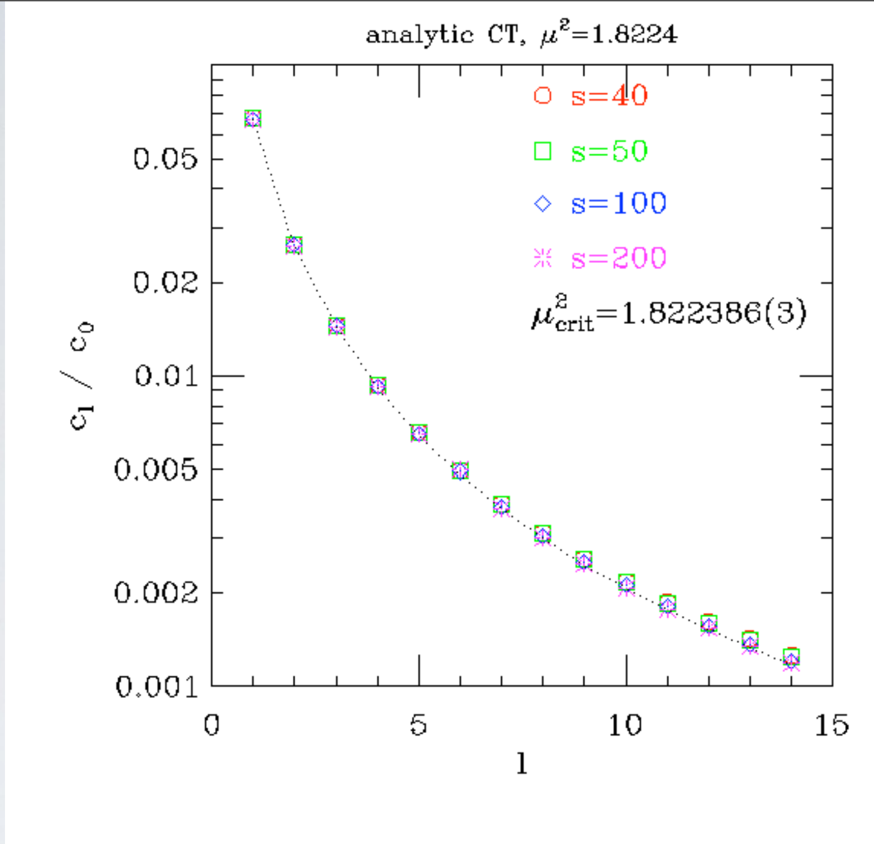


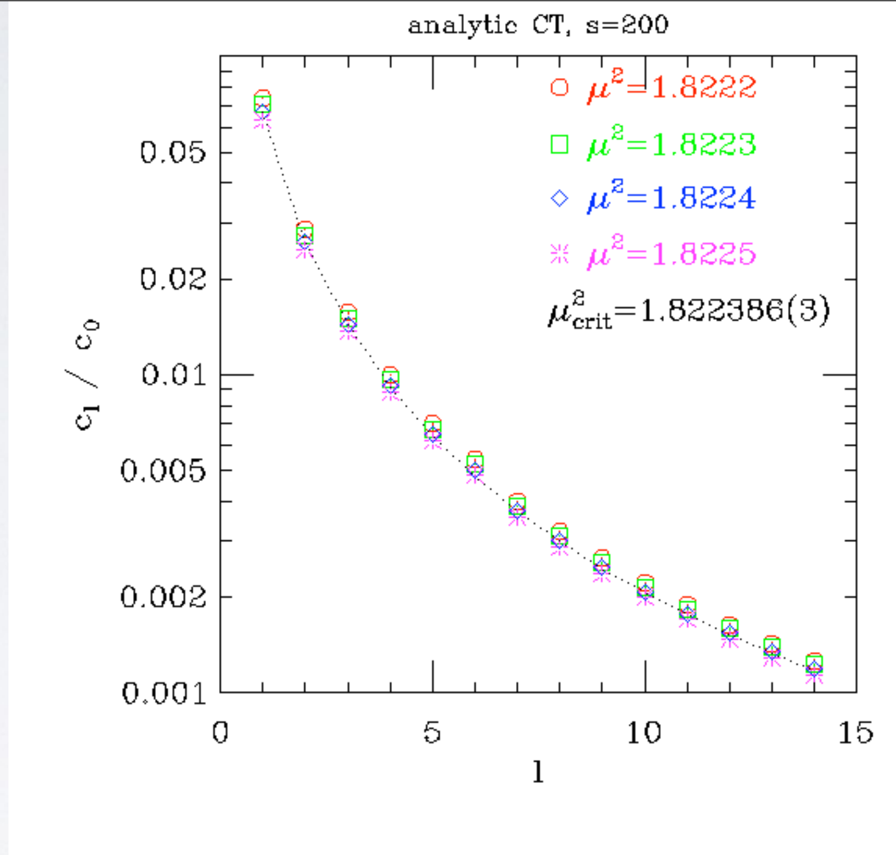
BINDER CUMMULANT



Fits: Uc = 0.5661(2) vs 0.567336

nu = 0.978(25) vs 1



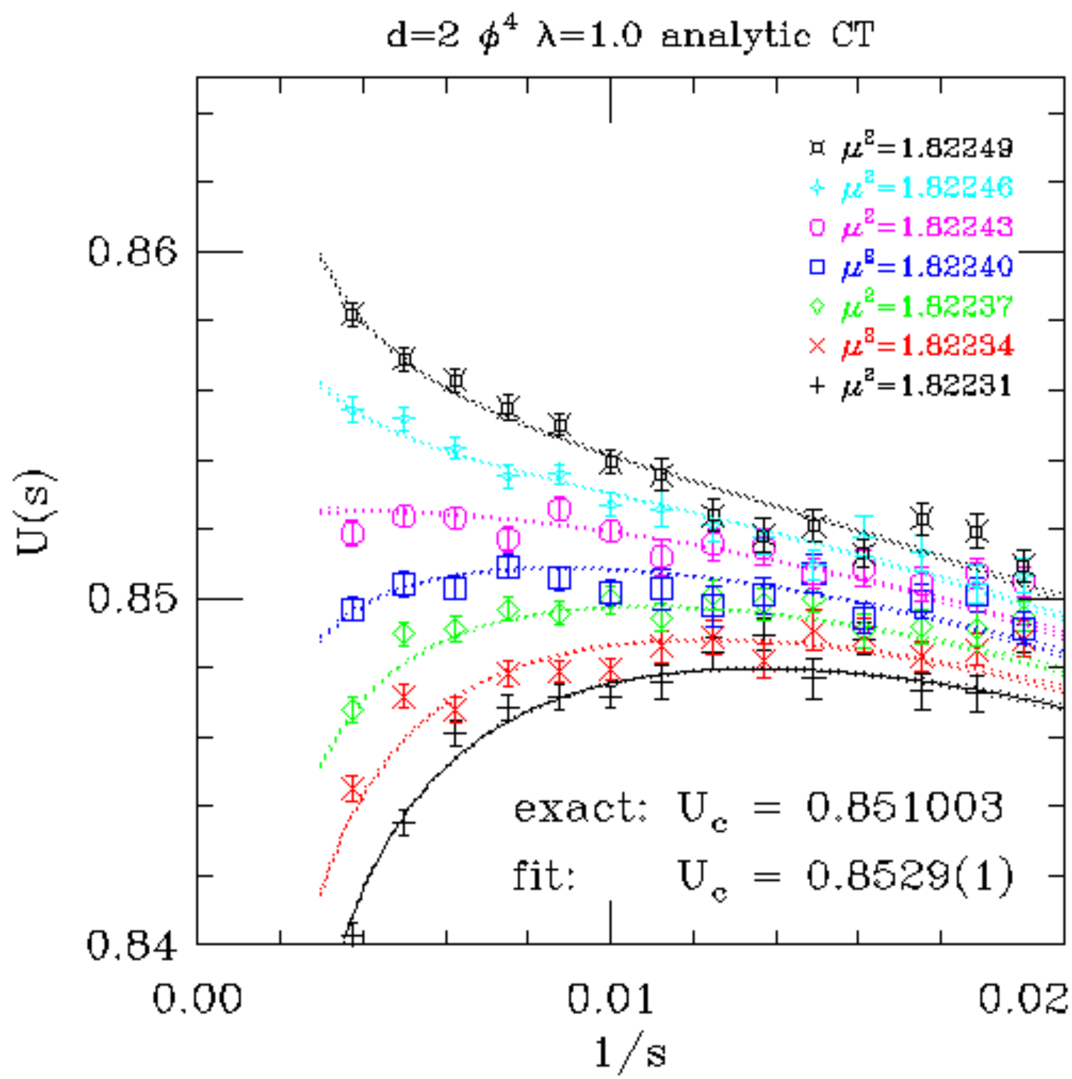


$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z) \qquad \Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

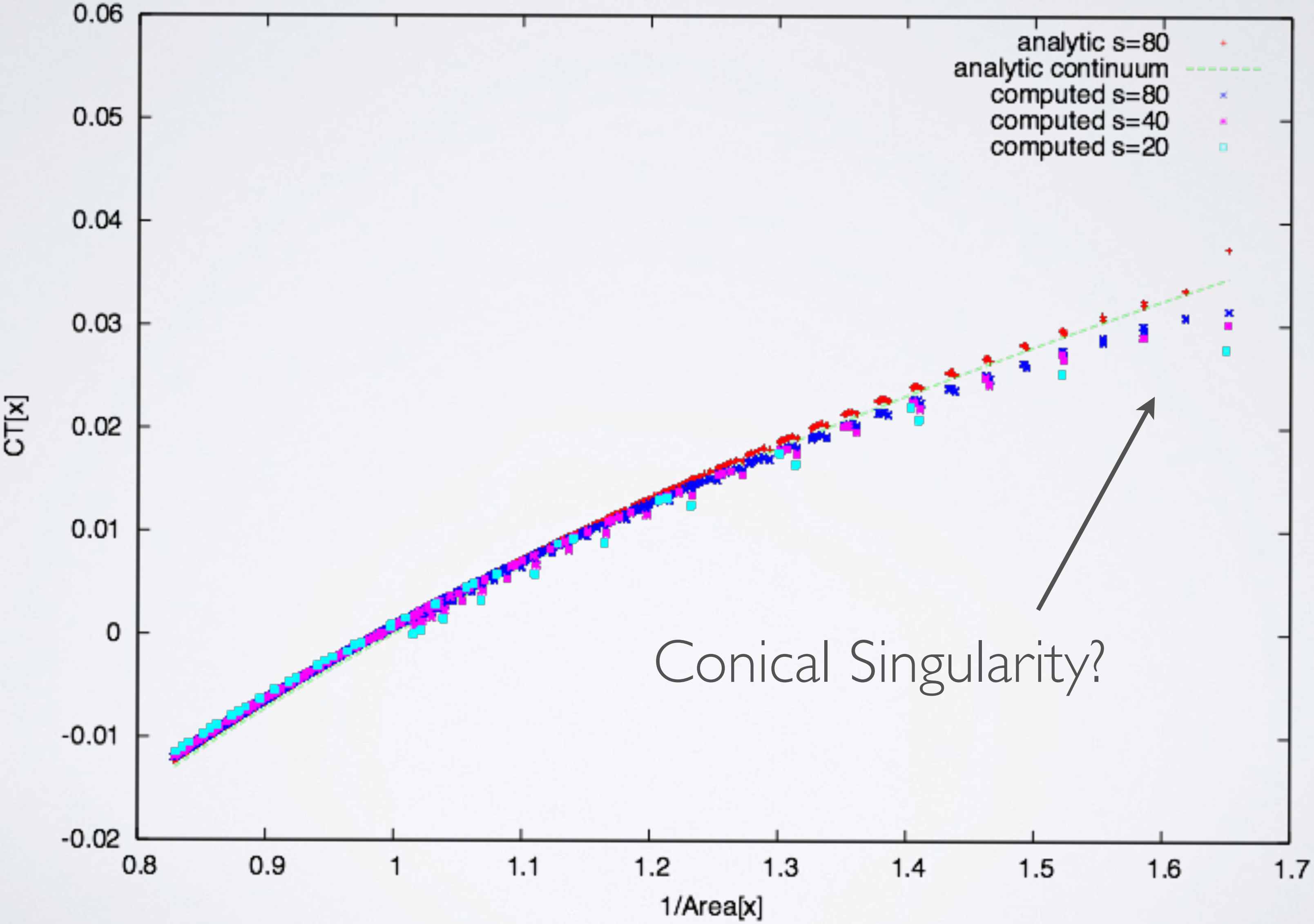
$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

Very fast cluster algorithm:
Brower,Tamayo ‘Embedded Dynamcis for phi 4th Theory” PRL 1989. Wolff single cluster + plus Improved Estimators etc

Binder Cummulant with Analytic Couter Term



ANALYTIC VS NUMERICAL CT



FERMIONS AND GAUGE FIELDS
"FEM" OR DISCRETE EXTERIOR CALCULUS

Fermions on smooth manifold:

$$\bar{\psi} e^{\mu}(x)(\partial_\mu - \omega_\mu)\psi$$

$$e^{\mu}(x) = e^{\mu}_{a}\gamma^{a}$$

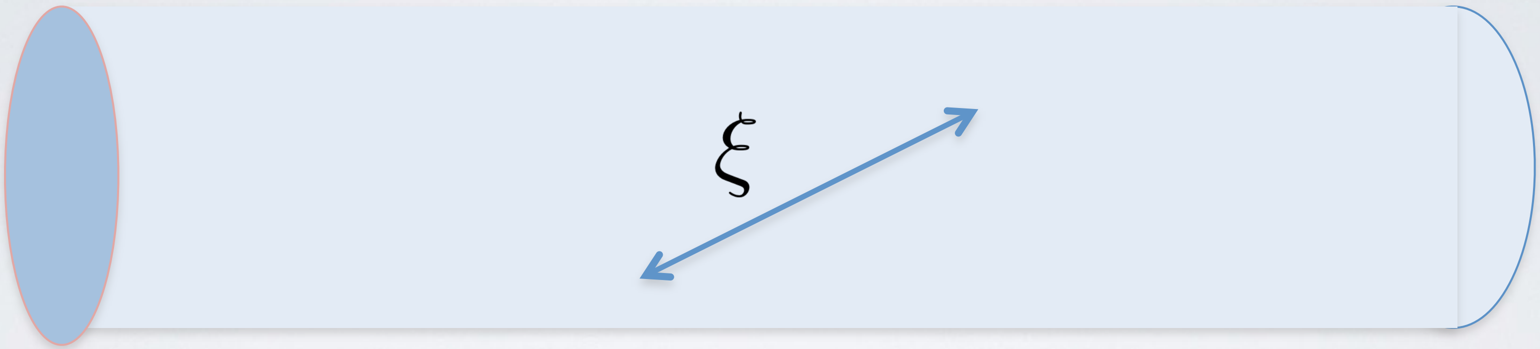
$$\omega_\mu(x) = \omega^{ab}_{\mu}\gamma_a\gamma_b \qquad g_{\mu\nu}(x) = e^{a}_{\mu}(x)e^{a}_{\nu}(x)$$

(1) Lattice Fermions are on simplicial curved manifolds must be done with great care: Spin connection does not always exist! (Doing c = 1/2 fermion on sphere)

(2) Compact gauge links can be using Nedelic/Whitney "edge" elements the spirit of Christ, Friedberg, Lee & G. 'tHooft

WHAT ABOUT RADIAL QUANTIZATION OF YANG MILLS? (EVEN

$\mathbb{R} \times \mathbb{T}^3$ vs $\mathbb{R} \times \mathbb{S}^3$



1. Short distances: Asymptotically free OPE
2. In confining phase: Hamiltonian evolution!
3. In Conformal Window: Dilatation evolution!
4. Relevant mass deformation

(Spectral flow: Anomalous Dimension ➔ Energies!)

EXTRAS

Radial Quantization: *Early History*

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. .... The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For d > 2 these correspond however, to curved spaces. The result is verified for the spherical model

Proof of one loop QFEM Counter Term:

Try: $$\delta m^2 = q(x)\log(1/a^2m^2) + c(x) + O(a^2)$$

- Step #1: FEM: Cea's lemma (no

$$||\phi(x) - \phi_h^{fem}(x)|| < ||\phi(x) - v_h(x)||$$

$$\text{where} \quad ||v(x)|| = \int_\Omega dx[\nabla v(x)\nabla v(x) + m^2 v^2(x)]$$

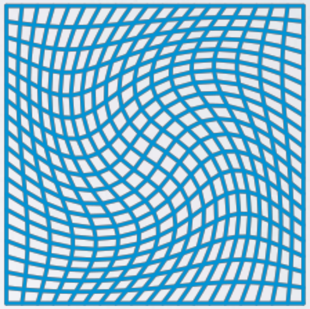
$$\implies q(x) = Q$$

- Step #2: Regge Gravity: diffeomorphisms

$$dzd\bar{z} = |f'(w)|^2 dwd\bar{w} \implies Qlog(|w|^2) = Q\log(|z|^2/\sqrt{g(z)})$$

WEYL VS CONFORMAL DIFFEOMORPHISMS

- Weyl (change the manifold) $$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)$$
- Diffeomorphism (change the co-ordinates) $$x \to \xi = f(x)$$
- Conformal Diffeomorphism (largest subgroup)
- Primary field: $$\phi(\xi) = b^{-\Delta}(x)\phi(x)$$



$$ds^2 = g_{\mu\nu}(\xi) d\xi^\mu d\xi^\nu$$

$$ds^2 = b^2(x) g_{\mu\nu}(x) dx^\mu dx^\nu$$

Theorem: $$|\xi_1 - \xi_2|^2 = b(x_1)|x_1 - x_2|^2 b(x_2)$$

SIMPLICIAL LATTICE INDUCES A DIFFEOMORPHISM

- We have found the counter term of the FEM Lagrangian
- Our sequence of simplicies induce a (conformal?) diffeomorphism
- BUT now need to find correct for primary operator.

$$\phi(x) \to \Phi(x) = b(x)^{-\Delta}\phi(x)$$

- This is now being implement (Stay tuned)
- CT for 3D are being computed and will be tested

SCALING VS FULL CONFORMAL SYMMETRY

- General Field Theory with Scale invariance and Poincare Invariance
- O(d) ==> O(d,1) (Isometries of AdS space)

$$x_\mu \to \lambda x_\mu \quad , \quad x_\mu \to \frac{x_\mu}{x^2}$$

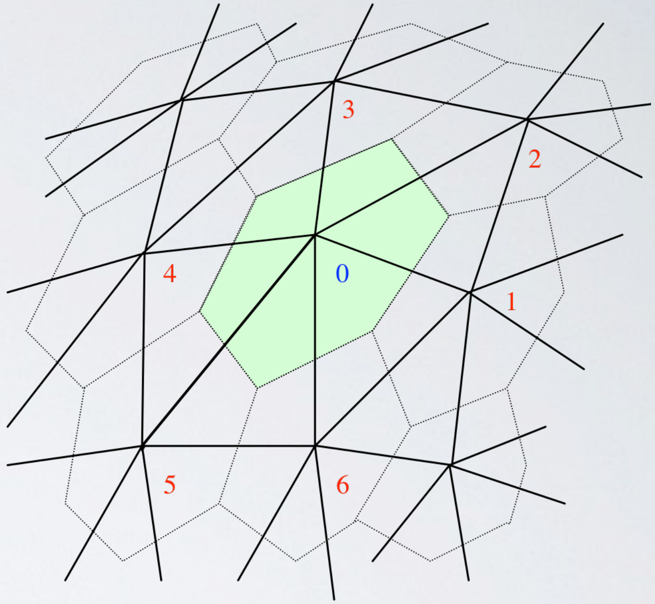
$$K_\mu : (inv \to trans \to inv)$$

$$[K_\mu, \mathcal{O}(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)$$

$$[D, P_\mu] = -iP_\mu \quad , \quad [D, K_\mu] = +iK_\mu \quad , \quad [K_\mu, P_\mu] = 2iD$$

Einstein Regge Curvature



$$\delta_v = 2\pi - \sum_{i \in V} \theta_i$$

$$\sum_v \delta_v = 2\pi\chi = 2\pi(F - E + V)$$

Could Optimize adaptive Delaney triangles on unit sphere

$$\int d^2x\sqrt{g}[\lambda - kR^2 + aR^2] \implies \sum_v A_v[\lambda - 2kR_v + aR_v^2]$$

$R_v = 2\delta_v / A_v$ flat triangles: $\delta_v = 4\pi / A_v$